

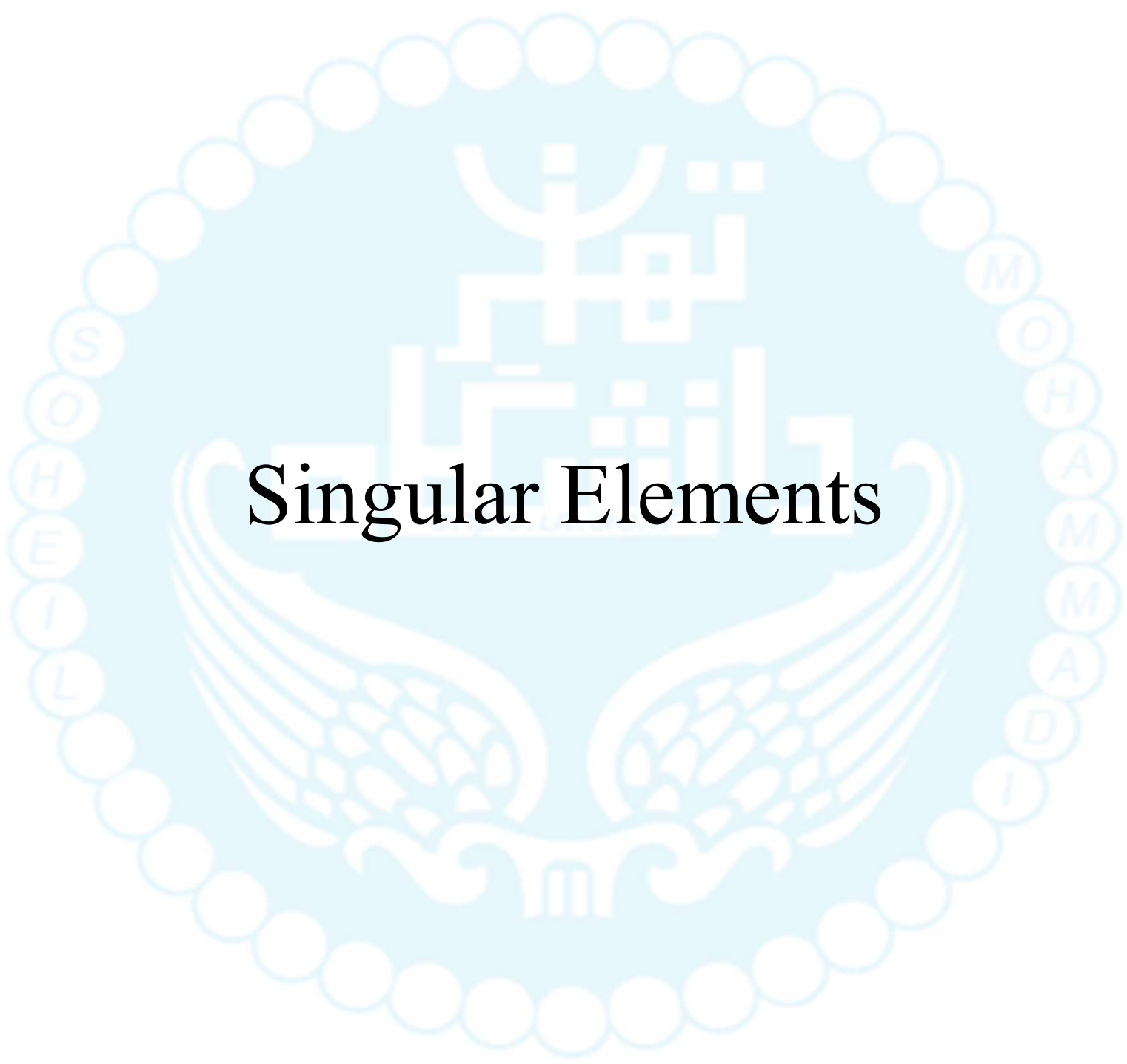


Fracture Mechanics

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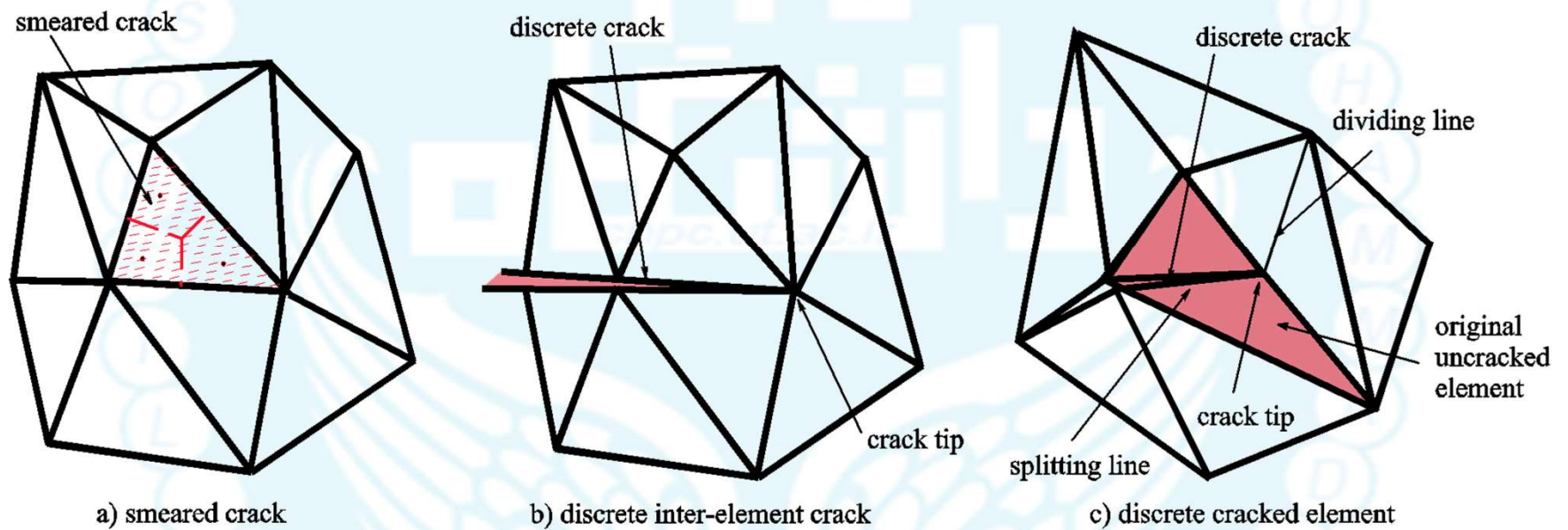
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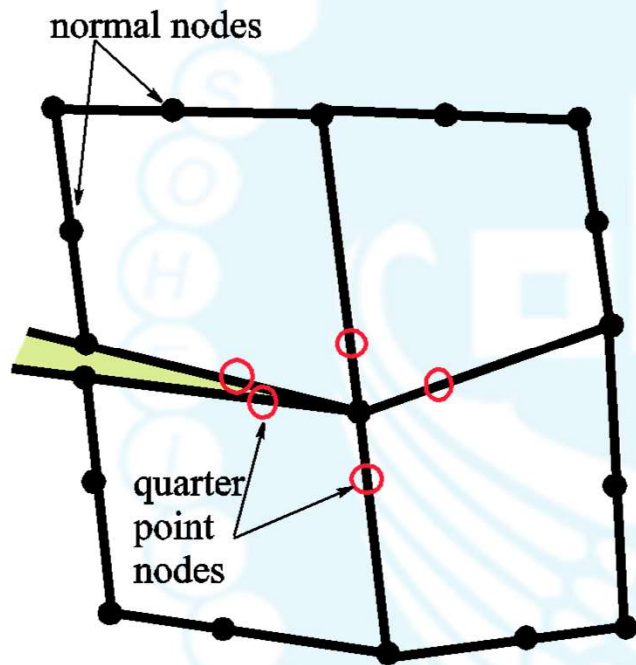


Singular Elements

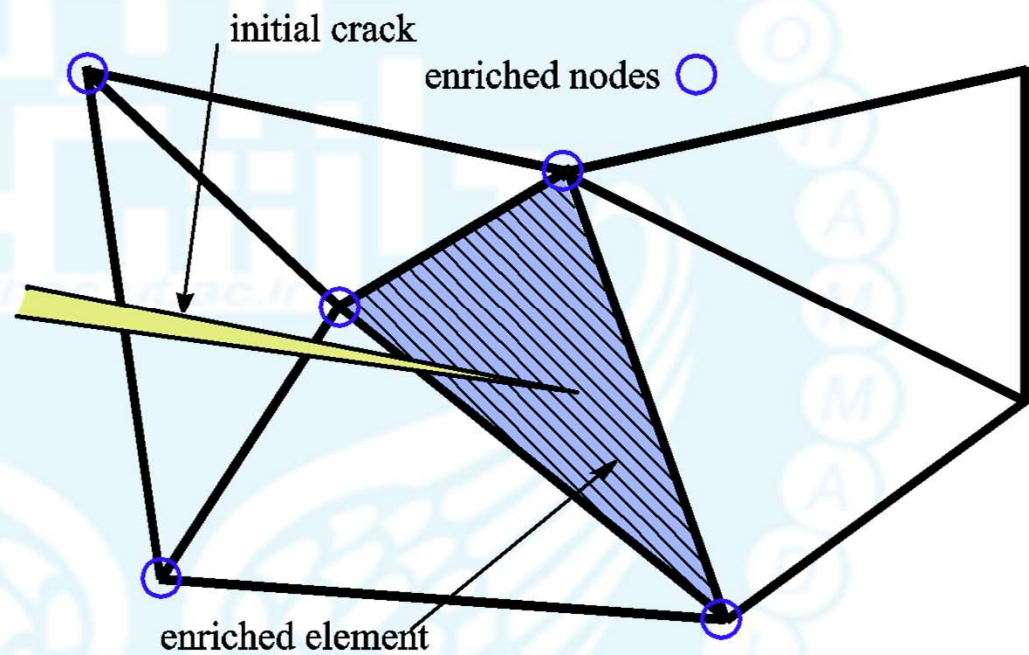
Classical crack modellings



Accurate crack modellings

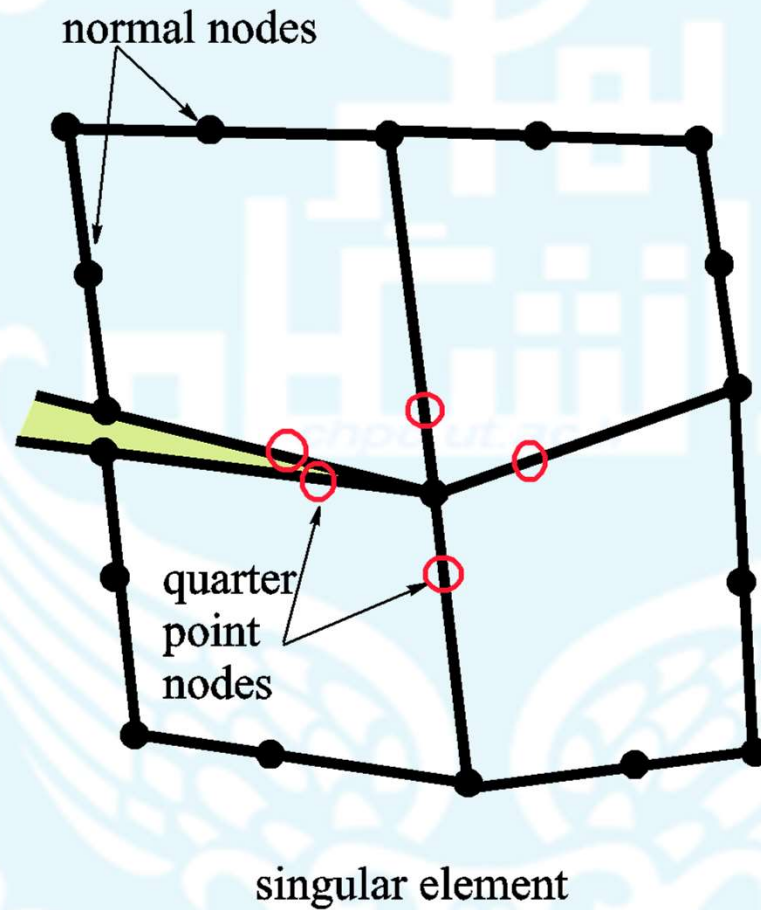


d) singular element



e) enriched element

Singular finite elements



Singular finite elements

In a major development, singular finite elements were developed for simulating crack tip singular fields. They provide major advantages; the model is simply constructed by moving the nearby mid-side nodes to the quarter points – absolutely no other changes in the finite element formulation are required.

The use of these elements has considerably upgraded the level of accuracy obtained by the finite element method for simulation of crack tip fields (Owen and Fawkes 1983).

Prior to the development of XFEM, singular elements have been the most popular approach for fracture analysis of structures.

Singular elements, however, lacked the capability of modelling discontinuity across a crack path. As a result, they had to be used with one of the other preceding methods to simulate the crack path.

One of the easiest and most powerful techniques for modelling a stress singularity is by quarter point singular elements.

Development of these singular elements had an enormous impact on wide range application of fracture mechanics by the finite element method while increasing the accuracy of various numerical analyses of LEFM.

It was first proposed by Barsoum (1974) and independently by Henshell and Shaw (1975), as they demonstrated that the singularity characteristic of LEFM can be obtained by two-dimensional 8-node isoparametric elements when the midside nodes near the crack tip are placed at the quarter points.

A direct consequence was the additional capability of an existing continuum based finite element code for modelling a stress singularity just by shifting the midside nodes of elements adjacent to the crack tip to their quarter point positions.

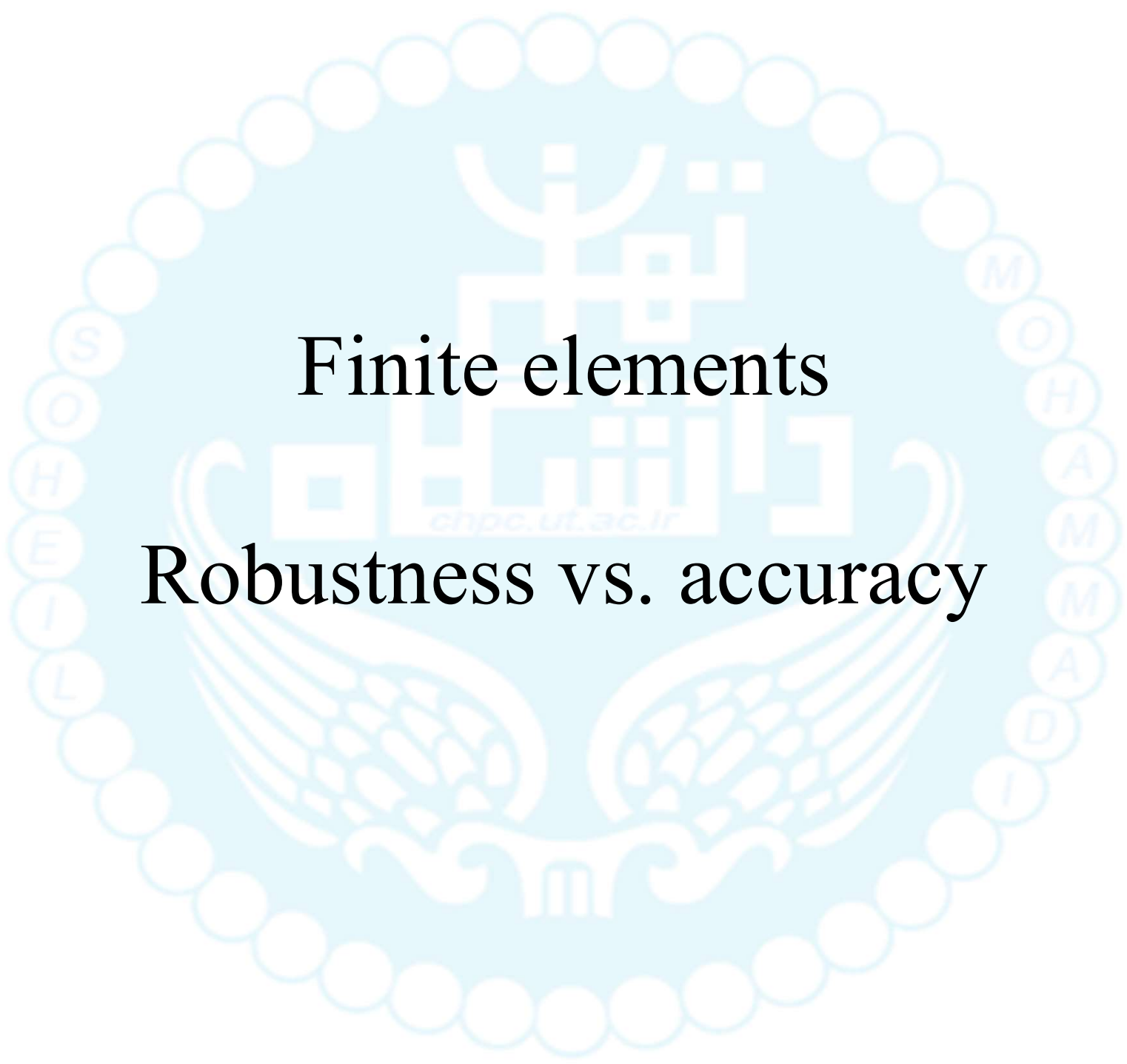
Hibbit (1977) and Ying (1982) studied the singularity of rectangular and triangular quarter elements and concluded that the singularity of a rectangular quarter point element is along the edges and diagonal only, whereas in a triangular element it occurs in all radial directions emanating from the crack tip.

In general quadrilateral singular elements, however, such a conclusion for edge-only-direction of singularity was put in doubt by a number of other researchers. Later, a complete study, including finite element implementation, related numerical issues and comparison of various singular crack tip elements were comprehensively presented by Owen and Fawkes (1983).

Singular finite elements

Interestingly, the powerful idea of singular finite elements has come from a major defect in finite element.

First, some major insight to finite element solution should be discussed.



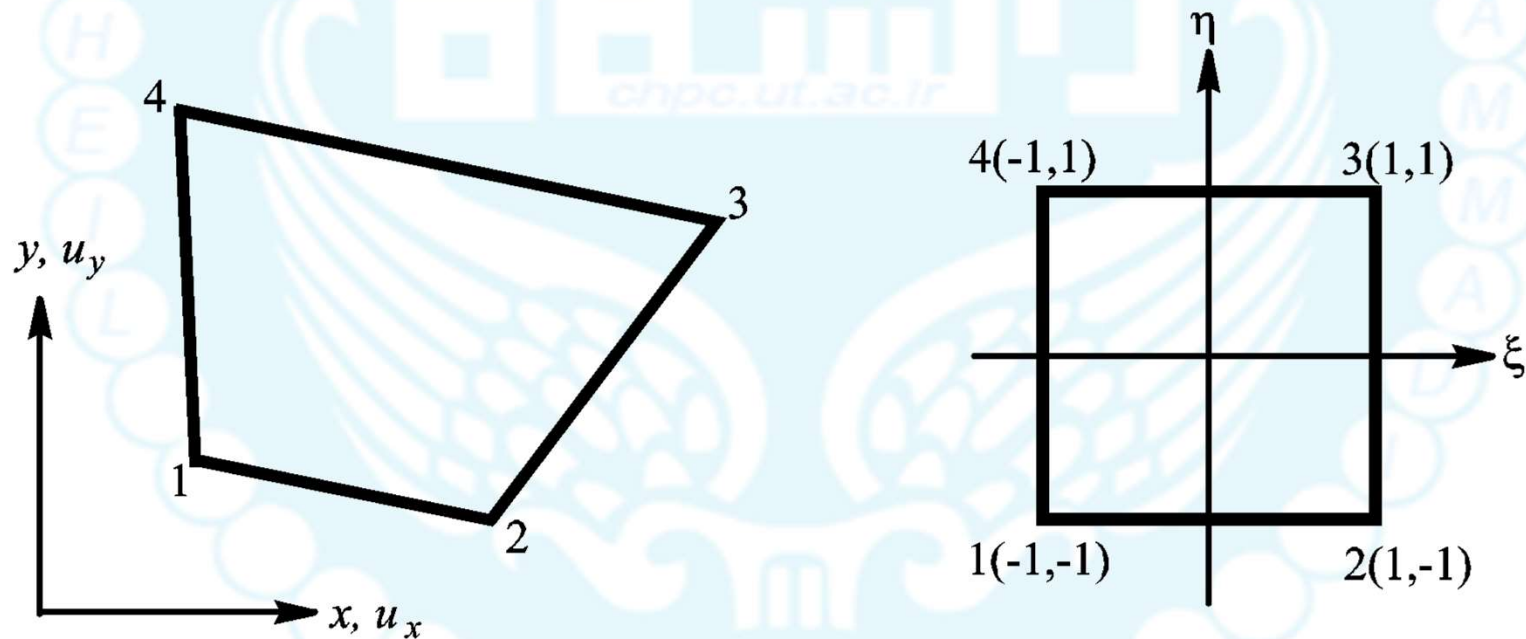
Finite elements

Robustness vs. accuracy

BASICS OF FEM

Isoparametric finite elements, a short review

Consider a domain in the state of equilibrium discretized by a four-node quadrilateral finite element mesh.



According to the finite element methodology, the coordinates $\mathbf{x}^T = (x, y)$ are interpolated from the nodal values $\bar{\mathbf{x}}^T = (\bar{x}, \bar{y})$:

$$\mathbf{x} = \sum_{j=1}^4 \mathbf{N}_j \bar{\mathbf{x}}_j$$

where $\mathbf{N}_j$ is the matrix of finite element shape functions,

$$\mathbf{N}_j = \begin{bmatrix} N_j & 0 \\ 0 & N_j \end{bmatrix}$$

In an isoparametric finite element representation, displacement fields are similarly interpolated from the nodal displacements nodal values:

$$\mathbf{u} = \sum_{j=1}^4 \mathbf{N}_j \bar{\mathbf{u}}_j$$

$$\mathbf{u}^T = (u_x, u_y)$$

$$\bar{\mathbf{u}}^T = (\bar{u}_x, \bar{u}_y)$$

The strain field is computed from:

$$\boldsymbol{\varepsilon} = \sum_{j=1}^4 \mathbf{B}_j \bar{\mathbf{u}}_j$$

where the matrix $\mathbf{B}_j$ is defined in terms of derivatives of the shape functions N_j ,

$$\mathbf{B}_j = \begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 \\ 0 & \frac{\partial N_j}{\partial y} \\ \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} \end{bmatrix}$$

and the chain rule is invoked to determine the coefficients of $\mathbf{B}_j$:

$$\begin{Bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{Bmatrix}$$

where $\mathbf{J}$ is the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad \mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

Finally, the stiffness matrix $\mathbf{K}_e$ of an element Ω_e can be determined from:

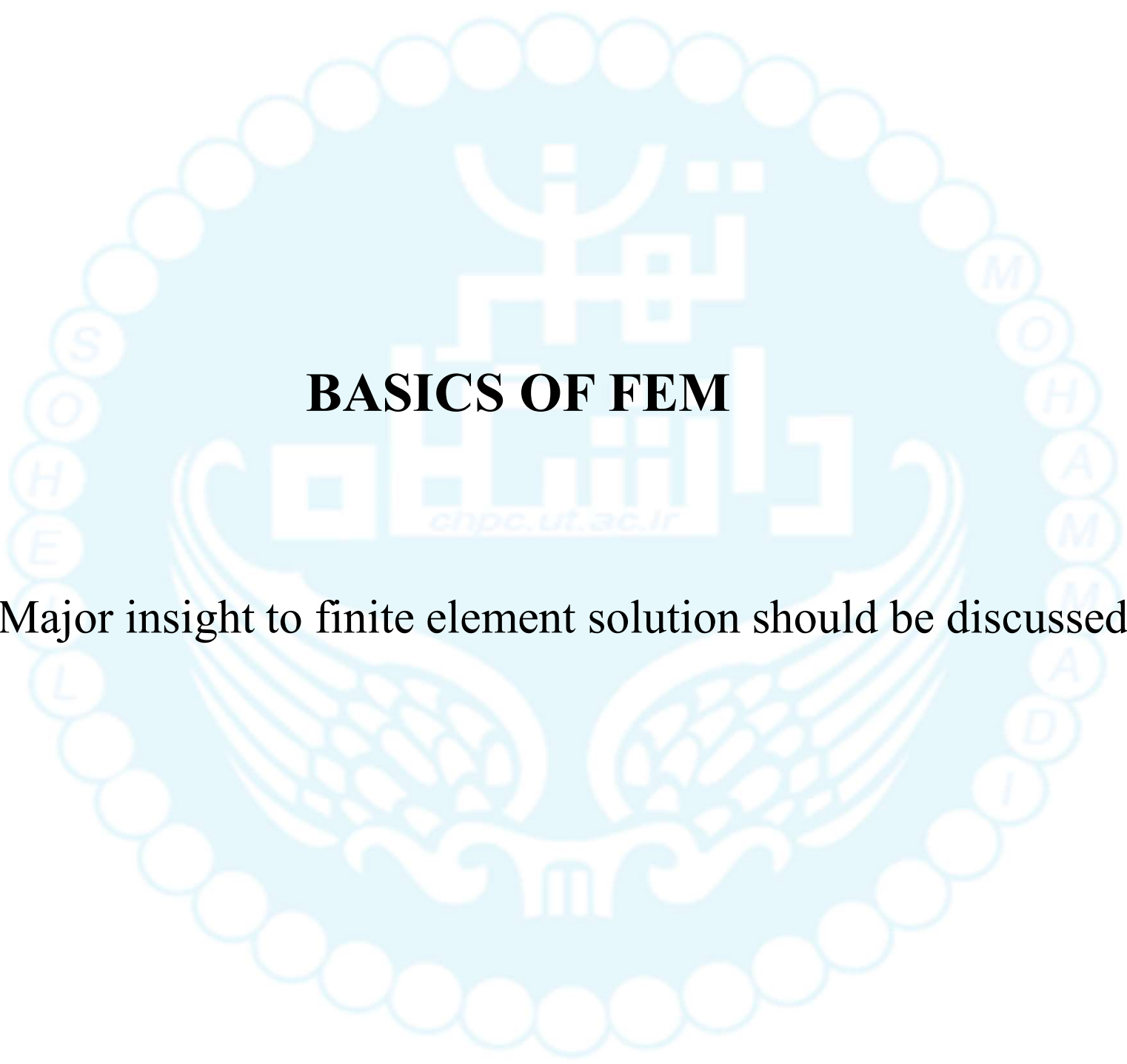
$$\mathbf{K}_e = \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega$$

where $\mathbf{D}$ is the material stress–strain or constitutive matrix:

$$\boldsymbol{\sigma} = \mathbf{D} \mathbf{B} \bar{\mathbf{u}}_i$$

and in the local curvilinear coordinates ξ, η :

$$\mathbf{K} = \int_{-1}^1 \int_{-1}^1 \mathbf{B}(\xi, \eta)^T \mathbf{D} \mathbf{B}(\xi, \eta) (\det \mathbf{J}) d\xi d\eta$$



BASICS OF FEM

Major insight to finite element solution should be discussed.

Basis functions/Shape functions

Interpolation inside a finite element is usually performed by shape functions N_i . They are known functions for various conventional finite elements.

$$u^h(x) = \mathbf{N}\bar{\mathbf{u}} = \sum_{i=1}^n N_i(x)\bar{u}_i$$

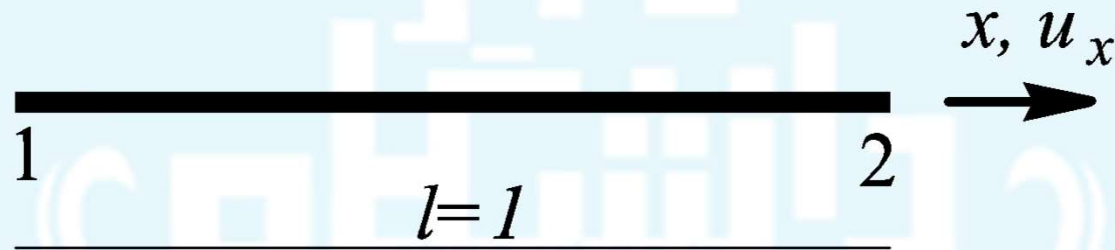
where u_i are the known nodal values.

Alternatively, the element interpolation may be defined in terms of the basis function P_i .

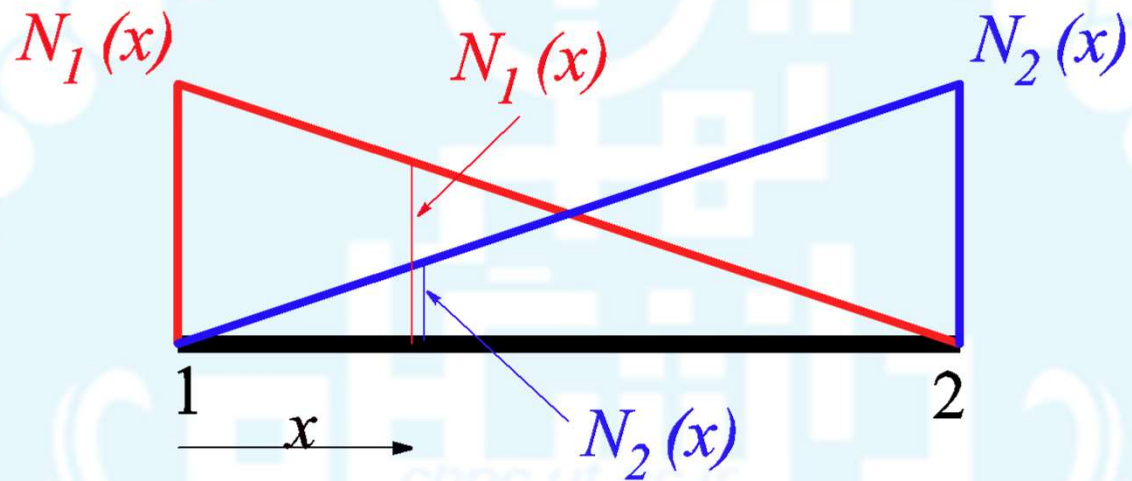
$$u^h(x) = \mathbf{P}\mathbf{a} = \sum_{i=1}^n P_i(x)a_i$$

where a_i are unknown constants.

To better discuss the subject, a 1D linear element is considered:



While, the shape functions may not be generally known, they are well defined in this simple problem:



$$N_1(x) = 1 - x$$

$$N_2(x) = x$$

$$u^h(x) = \sum_{i=1}^2 N_i(x) \bar{u}_i = N_1(x) \bar{u}_1 + N_2(x) \bar{u}_2$$

Now, if the basis function formulation is adopted, the element requires two basis function components; $1, x$:

$$P^T = \{1, x\}$$

$$u^h(x) = \sum_{i=1}^2 P_i(x) a_i = P_1(x) a_1 + P_2(x) a_2$$

$$\rightarrow u^h(x) = (1) a_1 + (x) a_2 = a_1 + a_2 x$$

The values of interpolation must match the nodal values:

$$\left. \begin{array}{l} \bar{u}_1 = u^h(x=0) = a_1 \\ \bar{u}_2 = u^h(x=1) = a_1 + a_2 \end{array} \right\} \rightarrow \left\{ \begin{array}{l} a_1 = \bar{u}_1 \\ a_2 = \bar{u}_2 - \bar{u}_1 \end{array} \right.$$

and the basis-function based finite element formulation::

$$\begin{aligned} u^h(x) &= a_1 + a_2 x = \bar{u}_1 + (\bar{u}_2 - \bar{u}_1)x \\ &= (1-x)\bar{u}_1 + (x)\bar{u}_2 \\ &= \underbrace{(1-x)\bar{u}_1}_{N_1(x)} + \underbrace{(x)\bar{u}_2}_{N_2(x)} \\ &= N_1(x)\bar{u}_1 + N_2(x)\bar{u}_2 \end{aligned}$$

which is similar to conventional finite element approximation; both formulations are equivalent.

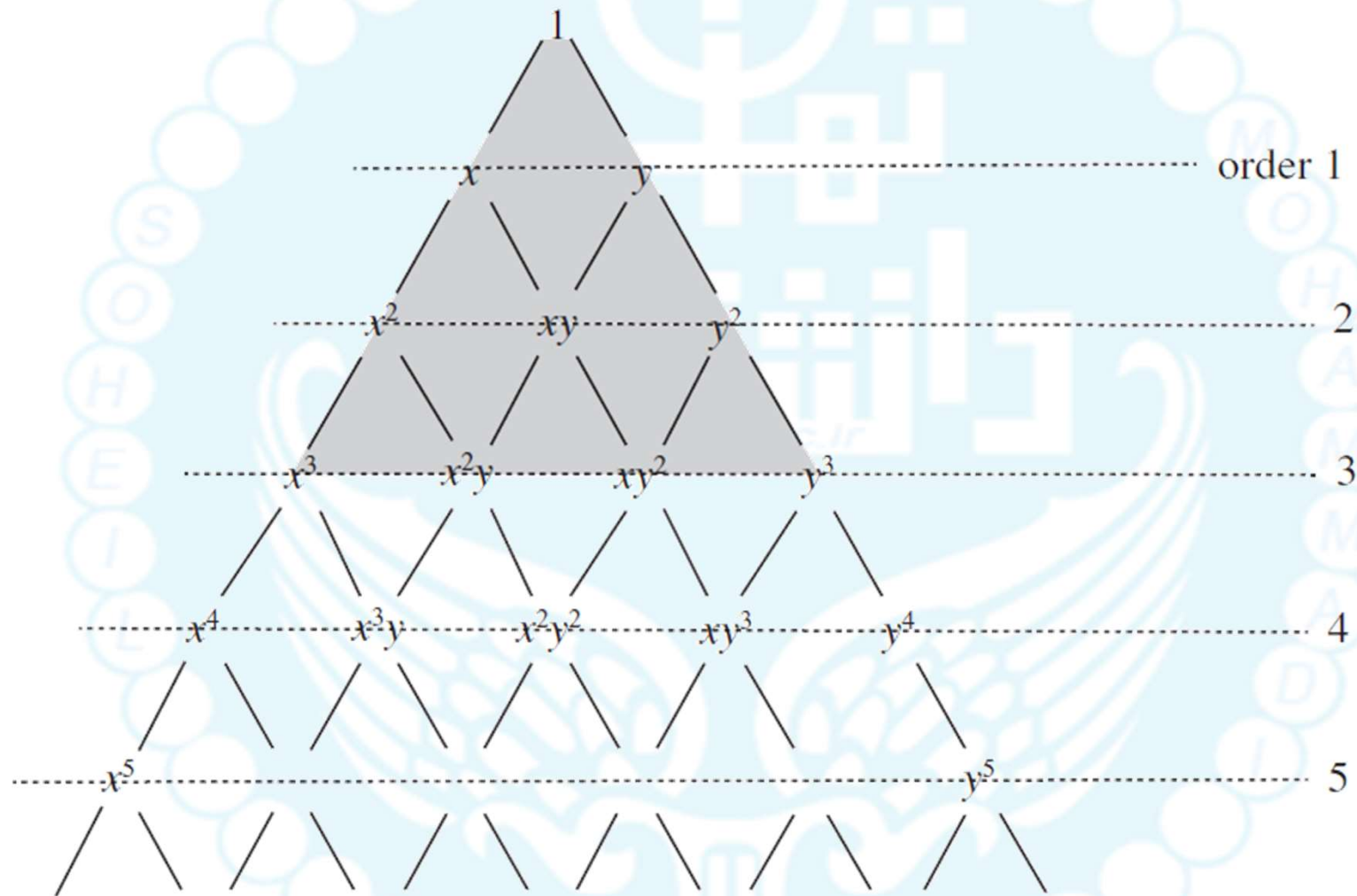
The basis function approach:

- It is a generalized approach for deriving the shape functions for any element, based on the selected set of basis function components.
- Many basic characteristics of the finite element formulations can better be studied by examining the basis function formulation.

Requirements for basis functions:

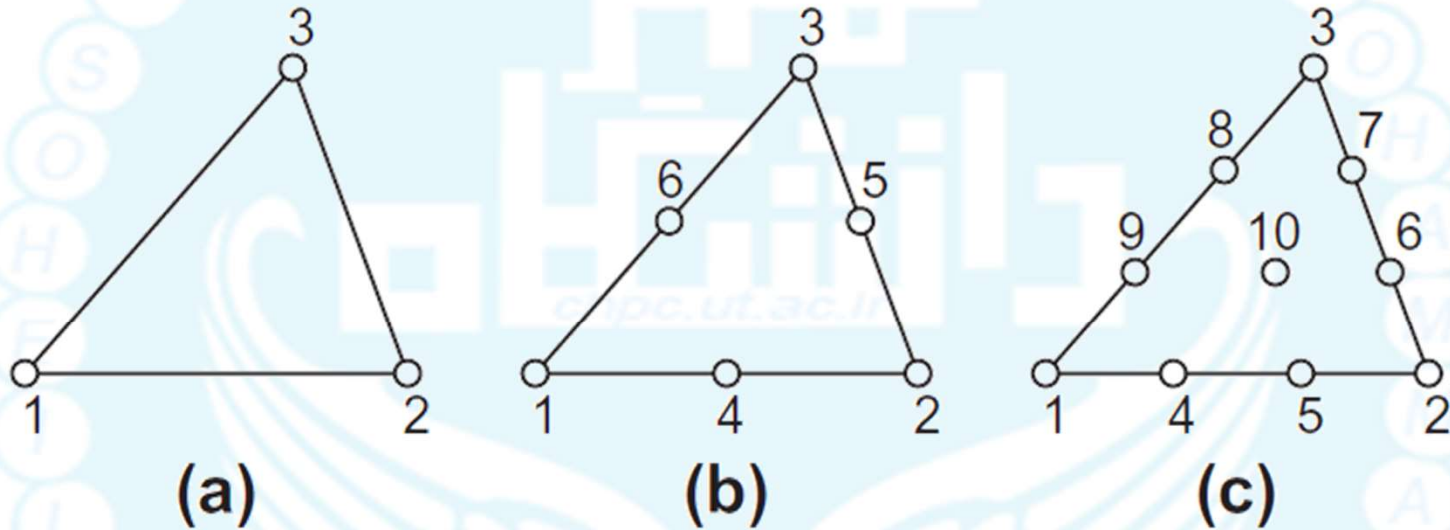
- Objectivity; invariant to coordinate rotation.
- Rigid body; the constant term.
- Completeness ?

Pascal triangle



The Pascal triangle (cubic expansion shaded—10 terms).

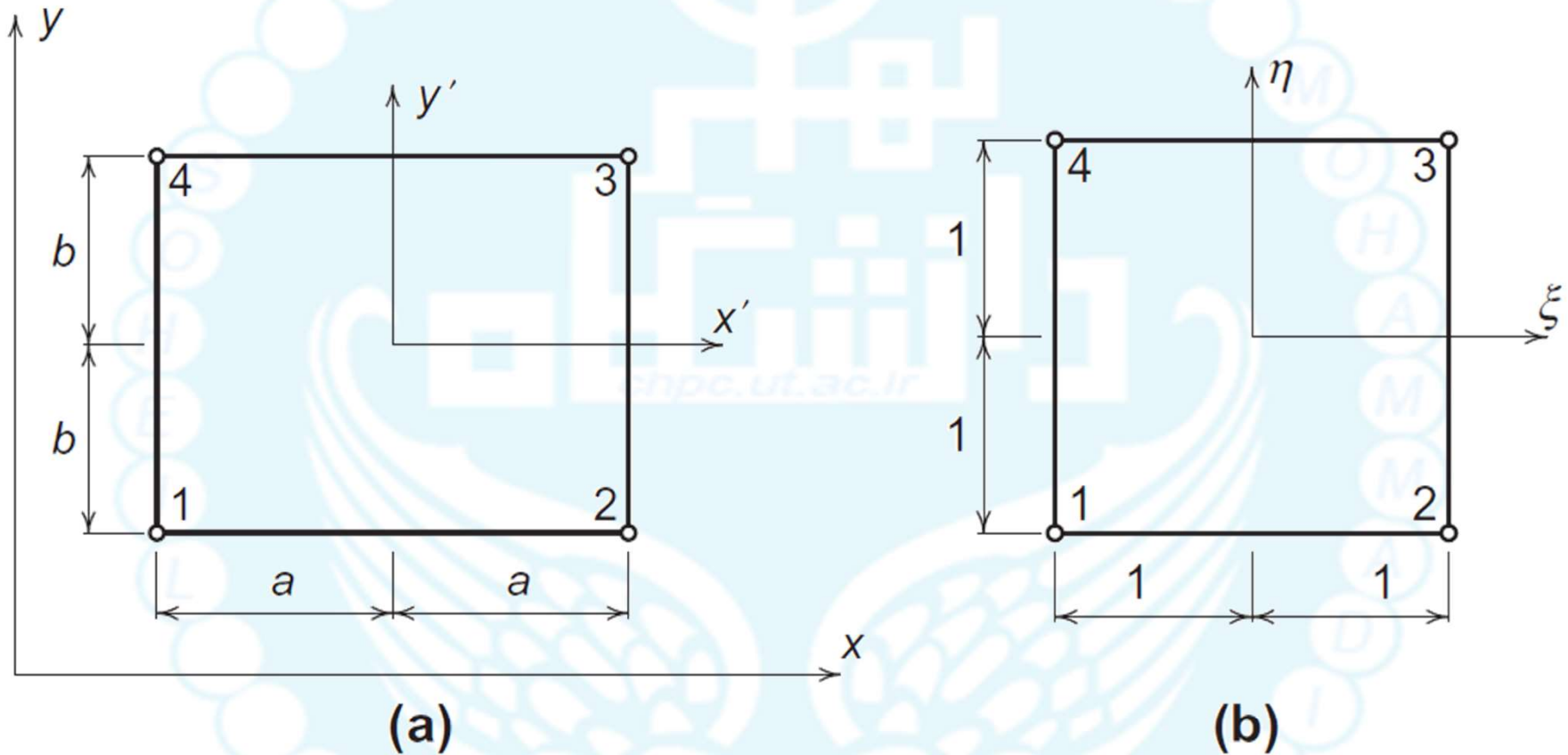
Higher order triangular elements



Triangular element family: (a) linear, (b) quadratic, and (c) cubic.



Quadrilateral with four nodes



Rectangular element (a) global coordinates and (b) parent coordinates.

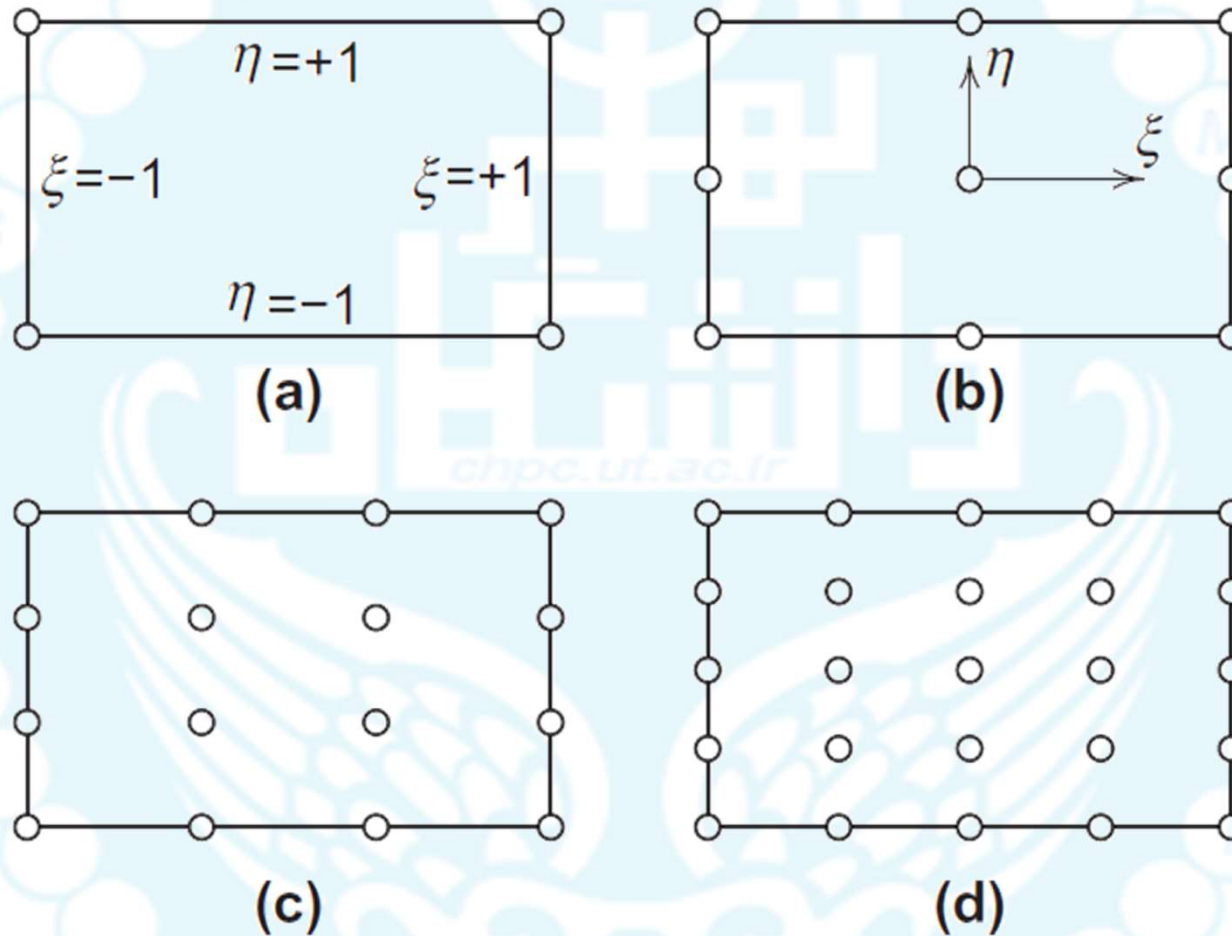
Quadrilateral with four nodes

$$N_1 = \frac{1}{4} \left(1 - \frac{x'}{a}\right) \left(1 - \frac{y'}{b}\right), \quad N_2 = \frac{1}{4} \left(1 + \frac{x'}{a}\right) \left(1 - \frac{y'}{b}\right)$$
$$N_3 = \frac{1}{4} \left(1 + \frac{x'}{a}\right) \left(1 + \frac{y'}{b}\right), \quad N_4 = \frac{1}{4} \left(1 - \frac{x'}{a}\right) \left(1 + \frac{y'}{b}\right)$$

$$\xi = \frac{x'}{a} \quad \text{and} \quad \eta = \frac{y'}{b} \quad -1 \leq \xi, \eta \leq 1$$

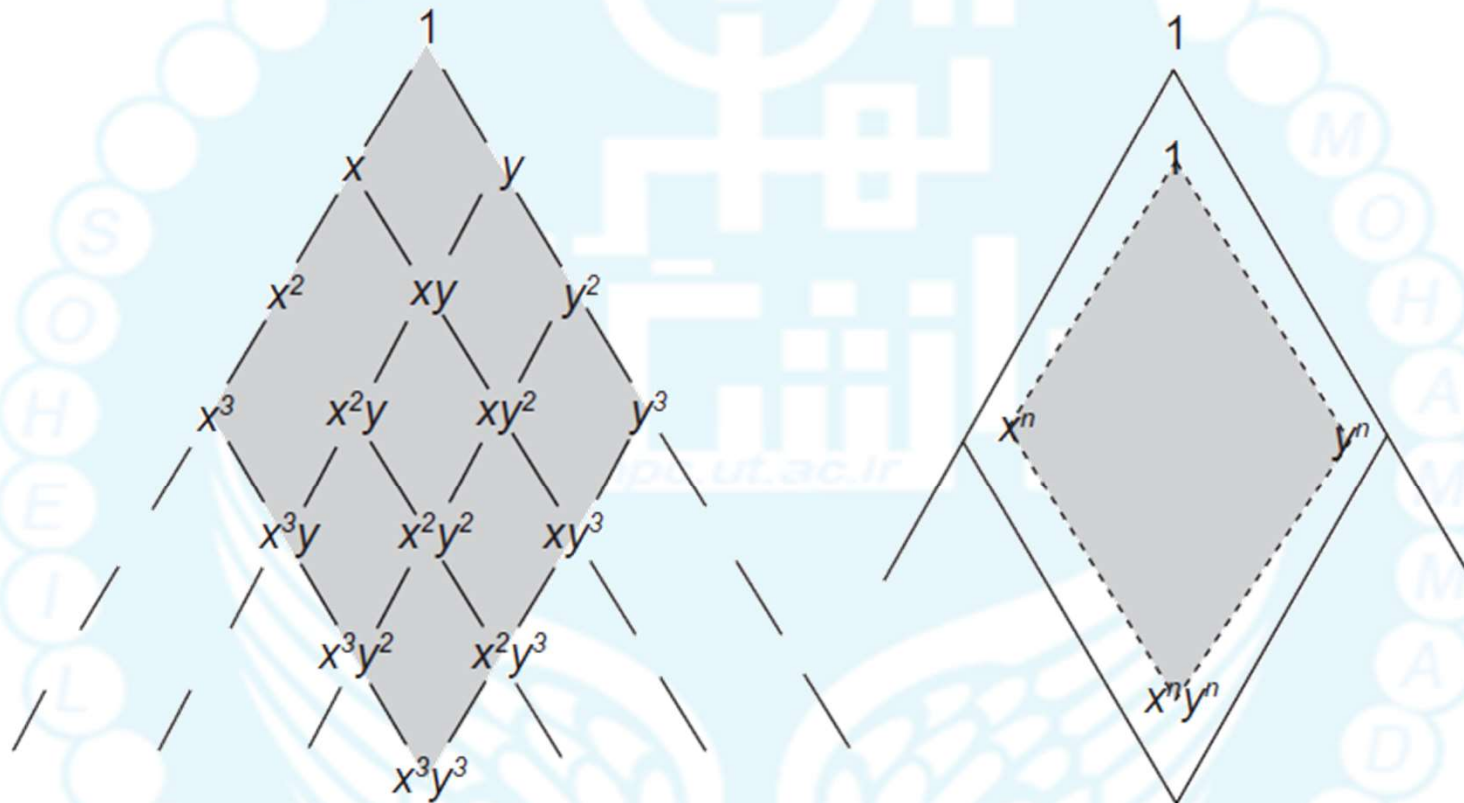
$$N_1 = \frac{1}{2}(1 - \xi) \cdot \frac{1}{2}(1 - \eta), \quad N_2 = \frac{1}{2}(1 + \xi) \cdot \frac{1}{2}(1 - \eta)$$
$$N_3 = \frac{1}{2}(1 + \xi) \cdot \frac{1}{2}(1 + \eta), \quad N_4 = \frac{1}{2}(1 - \xi) \cdot \frac{1}{2}(1 + \eta)$$

Lagrangian family of quadrilaterals



Four elements of the Lagrangian family:
(a) linear, (b) quadratic, (c) cubic, and (d) quartic.

Lagrangian family of quadrilaterals

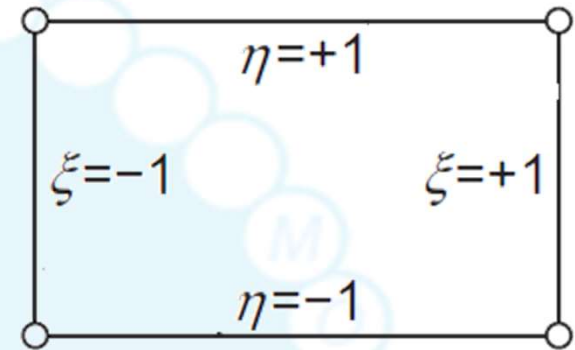


Terms generated by a Lagrangian expansion of order 3×3 . The shaded region shows complete polynomials of order 3 and n .

“Linear” element

$$N_a \equiv N_{IJ} = l_I^n(\xi) l_J^m(\eta)$$

$$N_a = \frac{1}{4}(1 + \xi_a \xi)(1 + \eta_a \eta)$$



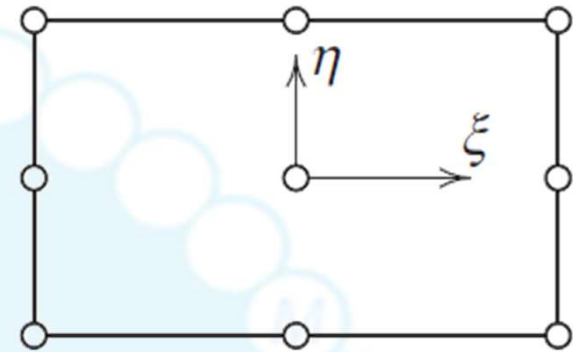
$$P^T = \{1, x, y, xy\}$$

$$P^T \neq \{1, x, y, x^2\}$$

$$P^T \neq \{1, x, y, x^2, xy, y^2\}$$

“Quadratic” Lagrangian element

Products of one-dimensional quadratic Lagrange interpolations yield the following:



Corner nodes:

$$N_a = \frac{1}{4}\xi\eta(\xi + \xi_a)(\eta + \eta_a)$$

Mid-side nodes:

$$\xi_a = 0, \quad N_a = \frac{1}{2}\eta(1 - \xi^2)(\eta + \eta_a)$$

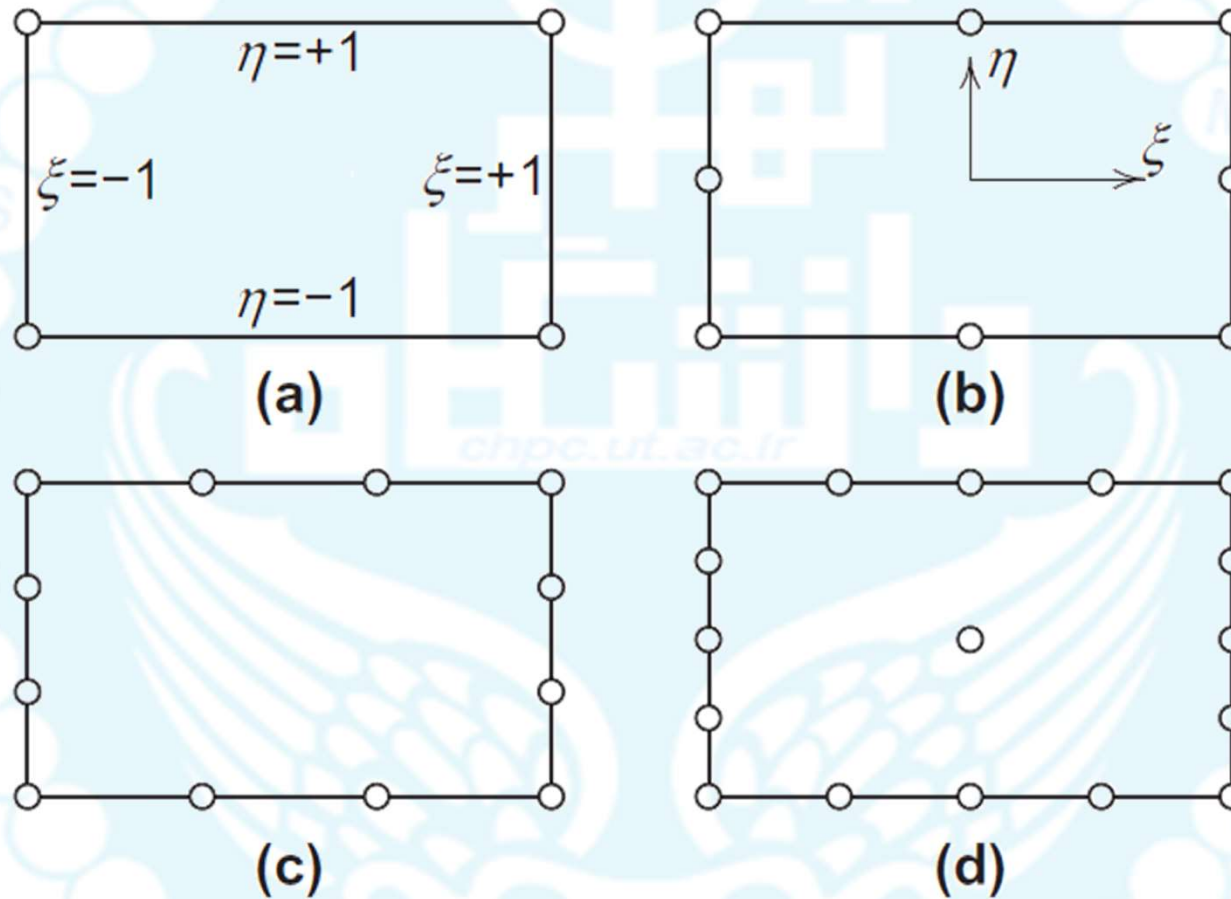
$$\eta_a = 0, \quad N_a = \frac{1}{2}\xi(\xi + \xi_a)(1 - \eta^2)$$

Center node:

$$N_a = (1 - \xi^2)(1 - \eta^2)$$

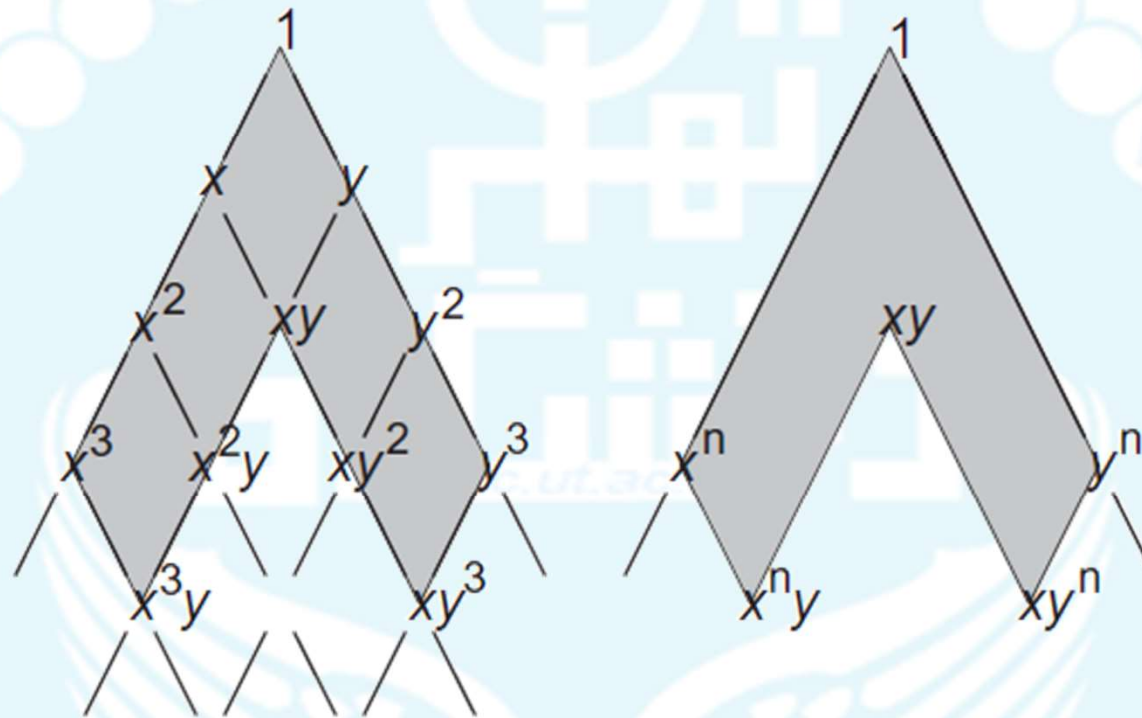
$$P^T = \{1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\}$$

Serendipity family of quadrilaterals



Four elements of the serendipity family:
(a) linear, (b) quadratic, (c) cubic, and (d) quartic.

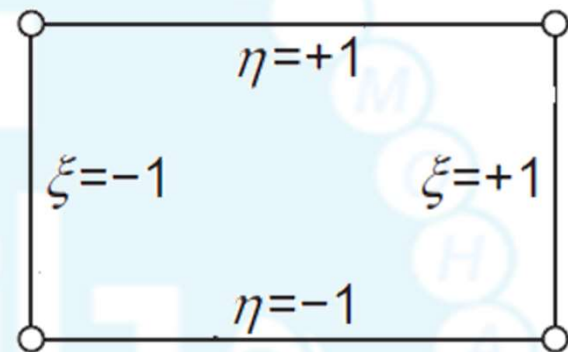
Serendipity family of quadrilaterals



Terms generated by edge-shaped functions in serendipity-type elements (3×3 and $n \times n$).

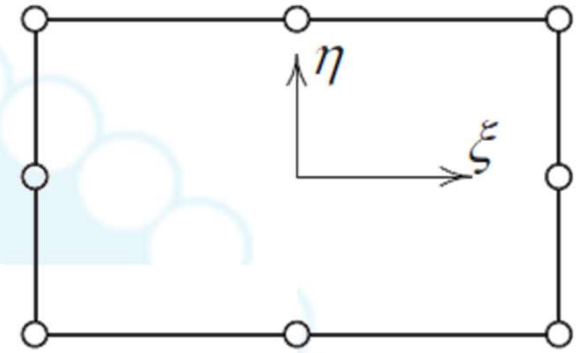
“Linear” element

$$N_a = \frac{1}{4}(1 + \xi_a \xi)(1 + \eta_a \eta)$$



$$P^T = \{1, x, y, xy\}$$

“Quadratic” Serendipity element



Corner nodes:

$$N_a = \frac{1}{4} \xi \eta (\xi + \xi_a)(\eta + \eta_a)$$

Mid-side nodes:

$$\xi_a = 0, \quad N_a = \frac{1}{2} \eta (1 - \xi^2)(\eta + \eta_a)$$

$$\eta_a = 0, \quad N_a = \frac{1}{2} \xi (\xi + \xi_a)(1 - \eta^2)$$

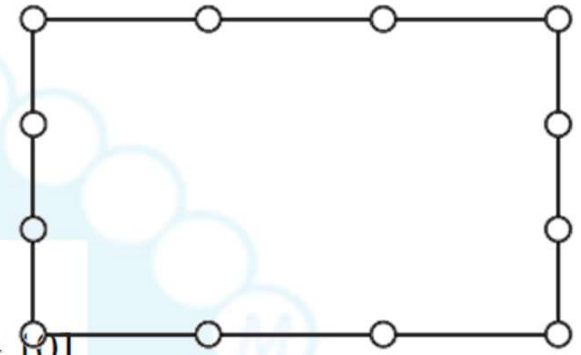
Center node:

$$N_a = (1 - \xi^2)(1 - \eta^2)$$

$$P^T = \{1, x, y, x^2, xy, y^2, x^2y, xy^2\}$$

$$P^T \neq \{1, x, y, x^2, xy, y^2, x^3, y^3\}$$

“Cubic” Serendipity element



Corner nodes:

$$N_a = \frac{1}{32}(1 + \xi_a \xi)(1 + \eta_a \eta)[9(\xi^2 + \eta^2) - 10]$$

Mid-side nodes:

$$\xi_a = \pm 1 \quad \text{and} \quad \eta_a = \pm \frac{1}{3}$$

$$N_a = \frac{9}{32}(1 + \xi_a \xi)(1 - \eta^2)(1 + 9\eta_a \eta)$$

and

$$\xi_a = \pm \frac{1}{3} \quad \text{and} \quad \eta_a = \pm 1$$

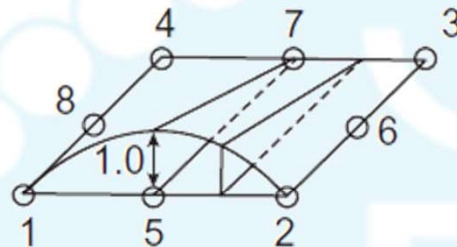
$$N_a = \frac{9}{32}(1 - \xi^2)(1 + 9\xi_a \xi)(1 + \eta_a \eta)$$

which all satisfy the requirement

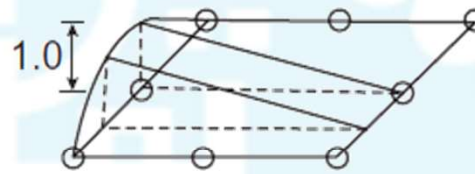
$$N_a(\xi_b, \eta_b) = \delta_{ab} = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases}$$

$$P^T = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3\}_{\beta_6}$$

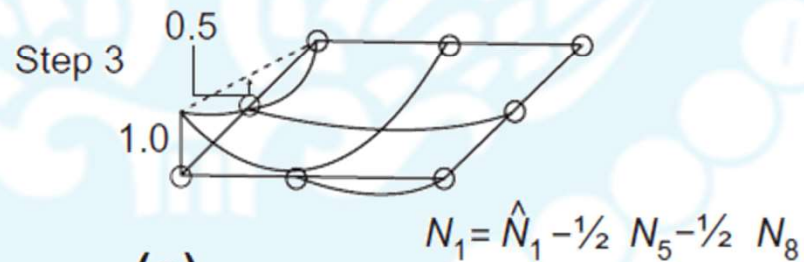
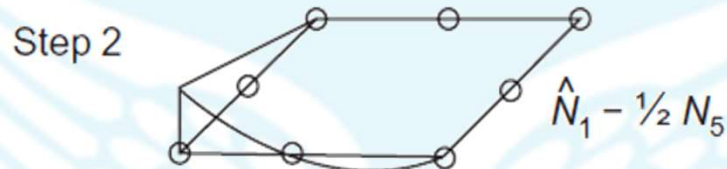
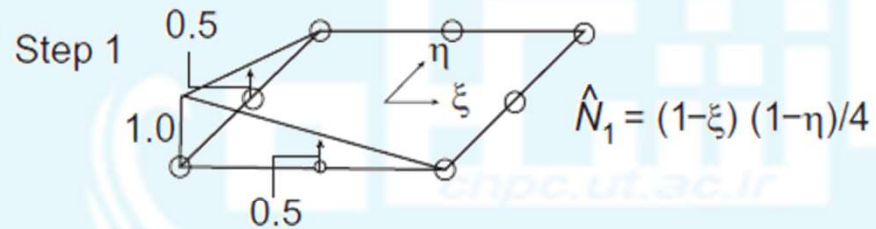
Serendipity family of quadrilaterals



(a) $N_5 = \frac{1}{2} (1 - \xi^2) (1 - \eta)$



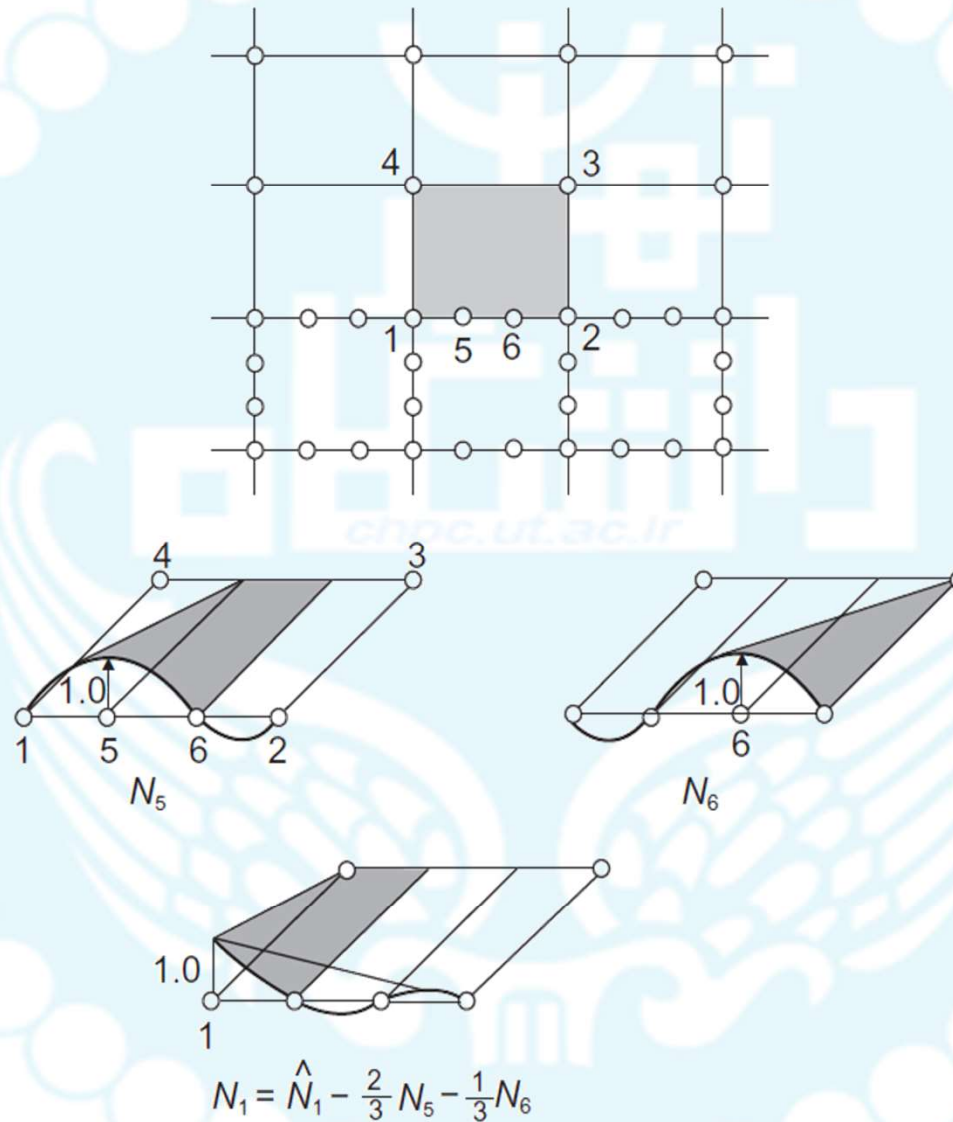
(b) $N_8 = \frac{1}{2} (1 - \xi) (1 - \eta^2)$



(c)

Systematic generation of “serendipity” shape functions.

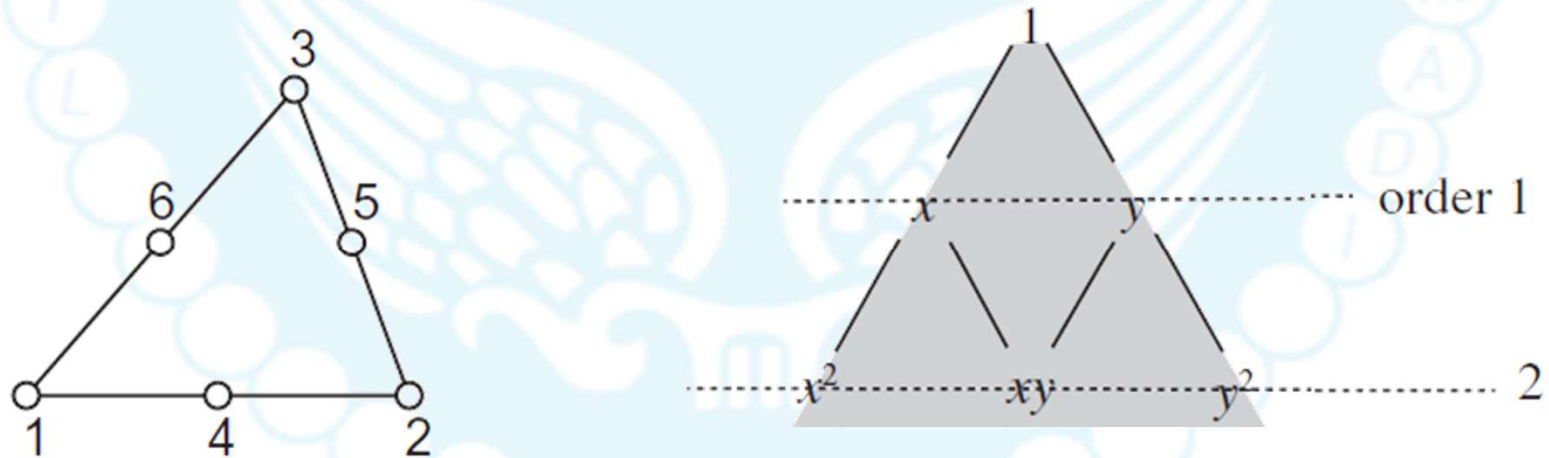
Modifications for a cubic/linear transition



Shape functions for a transition "serendipity" element, cubic/linear.

Requirements for basis functions:

- Objectivity; invariant to coordinate rotation.
- Rigid body; the constant term.
- Completeness ?



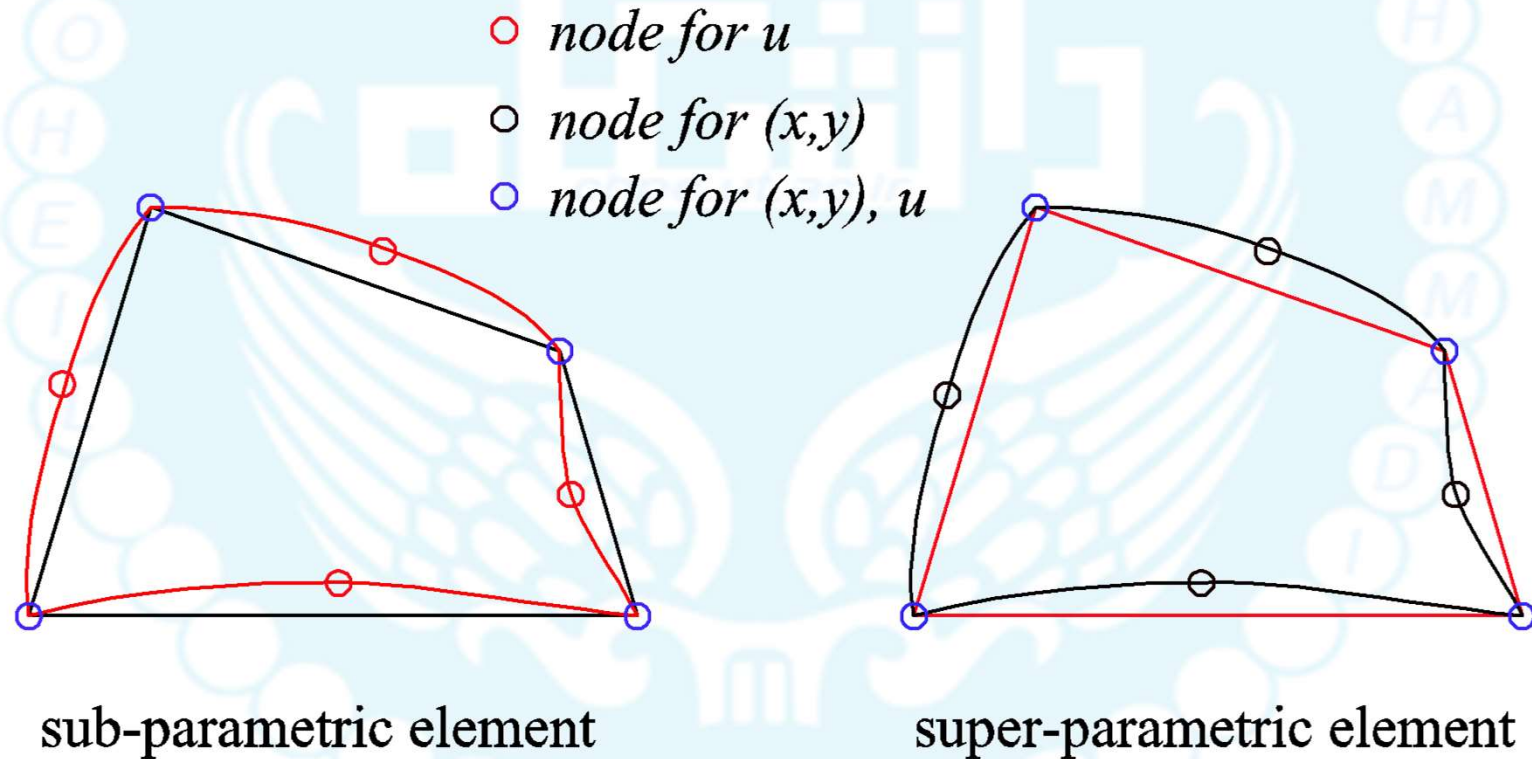




Interpolation failures

Sub/super parametric mapping

Both sub-parametric and super-parametric quadrilateral elements may also be used:



In sub-parametric elements:

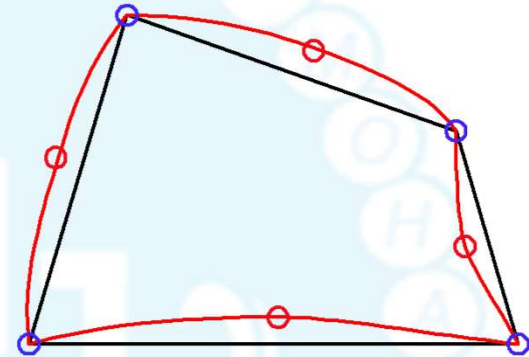
- The edges of elements are straight.
- For geometry:

$$P^x = [1, \xi, \eta, \xi\eta]$$

- For displacement field:

$$P^u = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2]$$

- *node for u*
- *node for (x,y)*
- *node for $(x,y), u$*



- Sub-parametric elements can reproduce linear strain states exactly. It is noted that iso-parametric 4-node elements cannot reproduce linear strain states exactly.

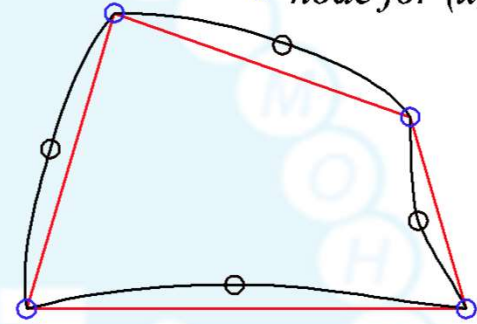
- The lowest order quadrilateral element that includes all terms of second order basis function, is the 9-node Lagrange element.
- As a result, it reproduces linear strain states exactly.
- This is an important advantage of the 9-node Lagrange element over the 8-node Serendipity element.
- The edge node locations must be effectively sub-parametric to better conform with straight edges and to avoid geometric inter-element penetration.

In super-parametric elements:

- The edges of elements may be curved.
- For geometry:

$$P^x = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2]$$

- node for u
- node for (x,y)
- node for $(x,y), u$



- For displacement field:

$$P^u = [1, \xi, \eta, \xi\eta]$$

- Super-parametric elements cannot reproduce
 - Constant strain states
 - Rigid body rotations

Interpolation failures

Difficulties may arise in interpolation procedures in special circumstances.

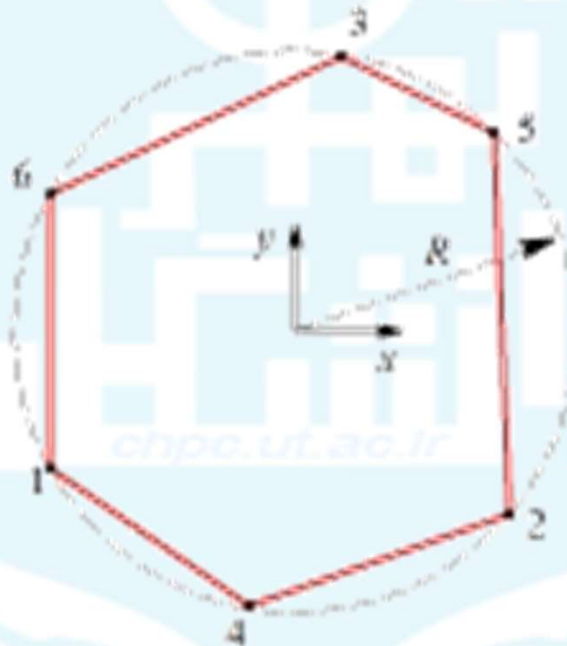
In a six-node triangle element, 6 basis functions govern the finite element interpolation:

$$\mathbf{P} = [1, x, y, x^2, xy, y^2]$$

If the six-node triangle is used to model a circle, the components of basis function $\mathbf{P}$ become dependent, and the interpolation procedure fails to determine the shape functions.

Not that this is an extreme case, which may not practically occur. It is only intended to show that general interpolation procedures may sometime fail in special circumstances.

When the six-node triangle is used to model a circle:

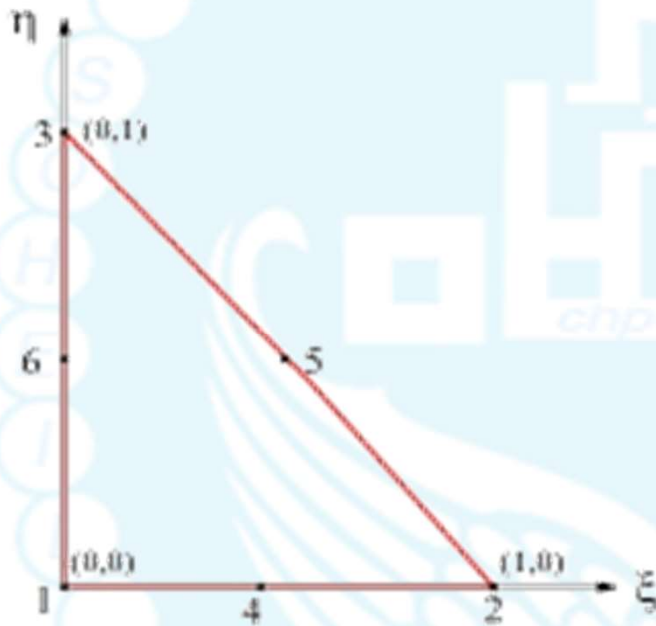


$$x^2 + y^2 = R^2 \rightarrow (P_4)_i + (P_6)_i = R^2 (P_1)_i$$

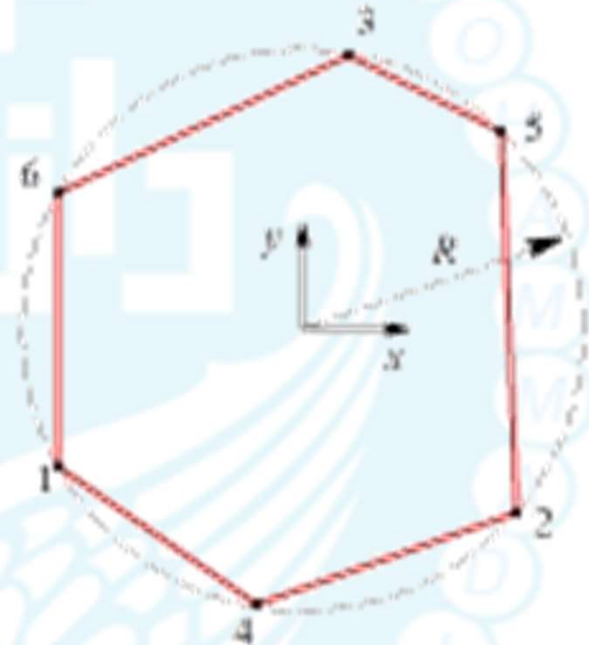
Therefore, a linear combination of the components of basis function $\mathbf{P}$ exists, which fails to determine the shape functions.

Homework:

Derive the shape functions of a 6-node element from the basis functions for a circular-arranged nodal points.



Parametric domain.



Physical domain.

$$P = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2]$$

$$P = [1, x, y, x^2, xy, y^2]$$

Note that in general circumstances, the following shape functions are expected to be obtained.

$$N_1 = 2(1 - \xi - \eta)\left(\frac{1}{2} - \xi - \eta\right)$$

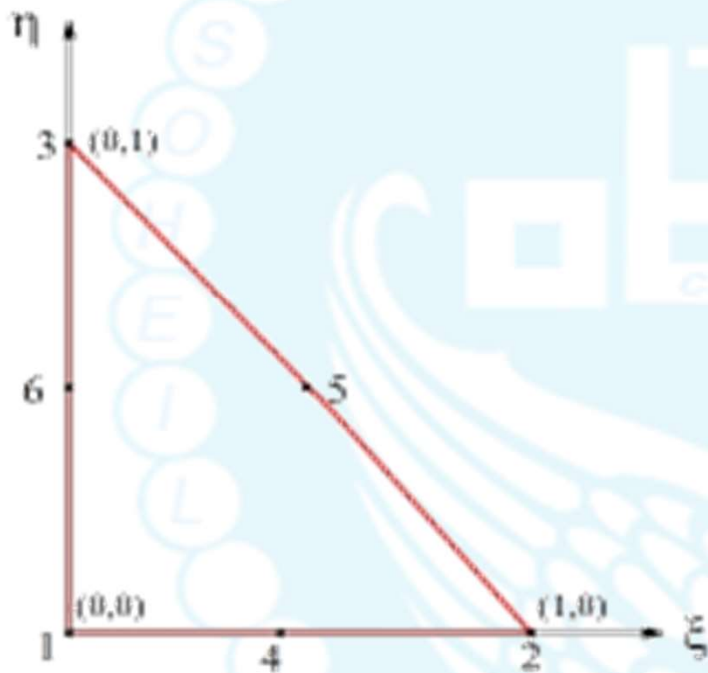
$$N_2 = 2\xi\left(\xi - \frac{1}{2}\right)$$

$$N_3 = 2\eta\left(\eta - \frac{1}{2}\right)$$

$$N_4 = 4\xi(1 - \xi - \eta)$$

$$N_5 = 4\xi\eta$$

$$N_6 = 4\eta(1 - \xi - \eta)$$





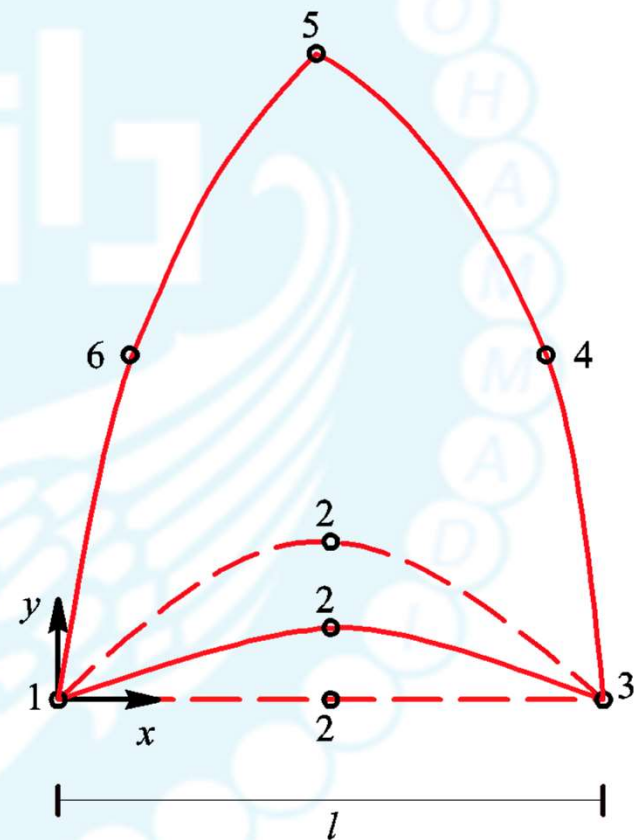
Continue as a Homework

Inter-element displacement continuity

Now, consider a curve edge with one edge node from a six-node triangle:

Now, assume the edge has a parabolic shape as:

$$y = \frac{cx}{l} \left(1 - \frac{x}{l} \right)$$



Then, for the six-node triangle, displacement along the curved edge can be derived.

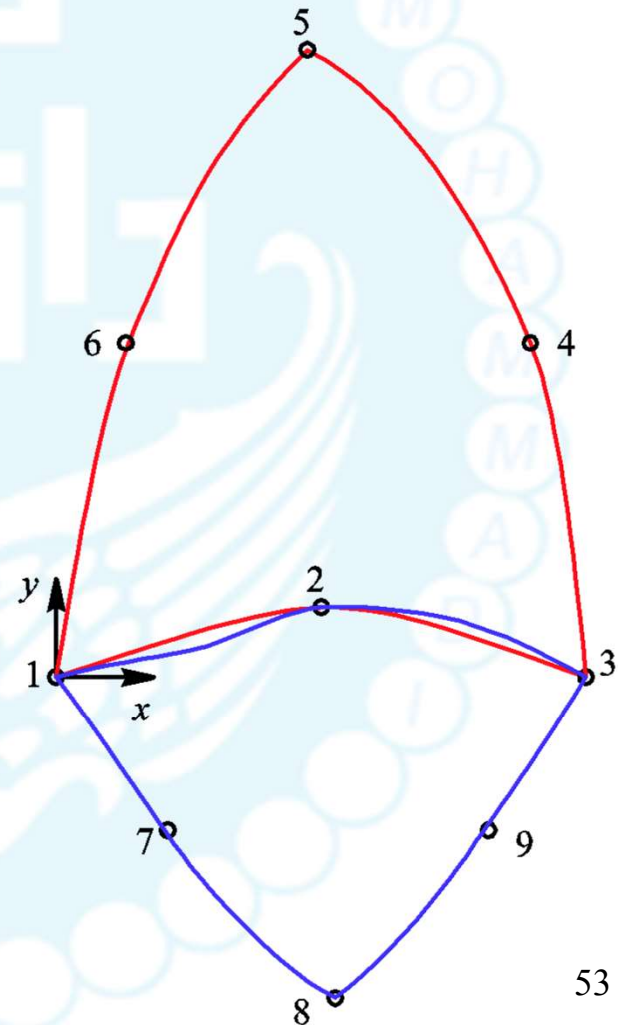
$$\begin{aligned} P &= [1, x, y, x^2, xy, y^2] \\ u &= a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \\ &= a_1 + a_2x + a_3 \left[\frac{cx}{l} \left(1 - \frac{x}{l} \right) \right] + a_4x^2 \\ &\quad + a_5x \left[\frac{cx}{l} \left(1 - \frac{x}{l} \right) \right] + a_6 \left[\frac{cx}{l} \left(1 - \frac{x}{l} \right) \right]^2 \\ &= A + Bx + Cx^2 + Dx^3 + Ex^4 \end{aligned}$$

which contains all powers of x up to the fourth order.

As a result, u contains five independent functions of x along the edge, whose coefficients clearly cannot be evaluated in terms of the values of u at only three nodes.

In fact, there will be no unique solution and different solutions may be obtained from adjacent elements; a clear sign of inter-element displacement discontinuity .

Parametric mapping will provide a way to achieve inter-element continuity for a larger range of shapes.



Interpolation failure beyond the range of basis function

Interpolation failure is an incorrect interpolation of a displacement field and its derivatives from its values at nodes (patch test).

Interpolation failure may occur naturally for higher-order functions beyond the range of basis functions (due to a finite number of basis functions):

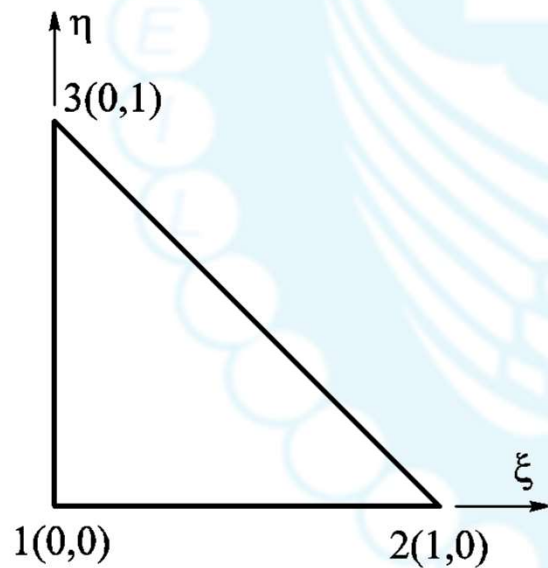
- For super-parametric mapping, correct interpolation of linear displacement field is not guaranteed.
- For iso-parametric mapping, correct interpolation of quadratic displacement is not guaranteed.

Locking phenomena may occur also occur in the form of excessive stiffness for a particular deformation state.

Interpolation failure beyond the range of basis function

There is usually a difference between the order of field an interpolation may sustain and the applied field.

To clarify the point, a 3-node element subjected to a quadratic displacement field is considered:



$$P = [1, \xi, \eta]$$

$$N_1 = 1 - \xi - \eta$$

$$N_2 = \xi$$

$$N_3 = \eta$$

Interpolation may be re-written in terms of the shape functions or basis functions, as:

$$u^h = N\bar{u} = PA\bar{u} = Pa$$

$$N = \{N_1 \quad N_2 \quad N_3\} = \{1 - \xi - \eta \quad \xi \quad \eta\}$$

$$P = \{1 \quad \xi \quad \eta\}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$u^h = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \bar{u} = \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix}$$

If the following quadratic displacement field is applied on the nodal points:

$$u = \xi^2 : \rightarrow \bar{u}_i = \xi_i^2 = \begin{Bmatrix} \xi_1^2 \\ \xi_2^2 \\ \xi_3^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\rightarrow a_j = A_{ji} \bar{u}_i = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\rightarrow u^h = Pa_j = \begin{Bmatrix} 1 & \xi & \eta \end{Bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \xi$$

Thus, within the 3-node element, an applied function $u = \xi^2$ can only be represented by $u^h = \xi$:

$$u = \xi^2 : \rightarrow u^h = \xi$$

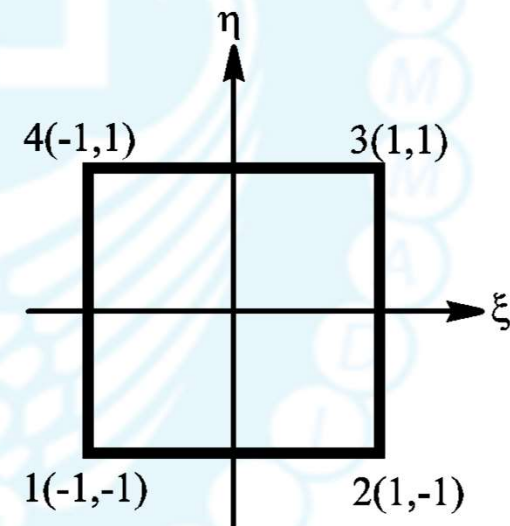
Similarly, for 4-node element: $P = \{1 \quad \xi \quad \eta \quad \xi\eta\}$

$$u = \xi^2 : \rightarrow u^h = 1$$

$$u = \xi^3 : \rightarrow u^h = \xi$$

$$u = \xi^2 \eta : \rightarrow u^h = \eta$$

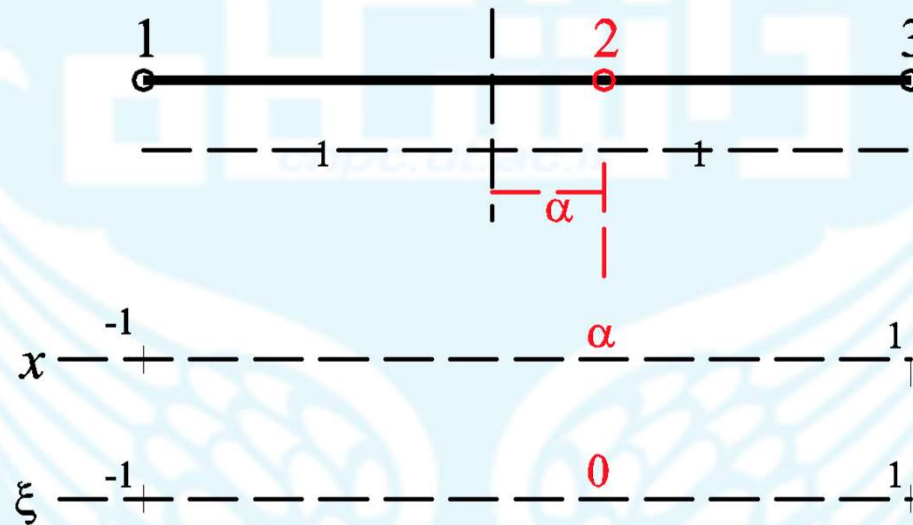
$$u = \xi^3 \eta : \rightarrow u^h = \xi \eta$$



Therefore, quite major differences are expected between any applied external field and the corresponding finite element internal field.⁵⁸

Interpolation in 3-node 1D element

In a 1D 3-node element, the center node may be located in between the two end nodes. Now, consider the center node is offset, and we wish to determine the accuracy of interpolation, by computing the accuracy of the axial strain field.



$$P = [1, \xi, \xi^2]$$

mapping between physical x and parametric ξ : $x = \xi + \alpha(1 - \xi^2)$

The constant strain state is clearly correctly evaluated.

Moreover, due to quadratic term ξ^2 in the basis function, it is expected that the quadratic displacement field, and so the linear strain field, should also be correctly evaluated. To examine, a quadratic displacement field is applied:

$$\begin{aligned} u^a = x^2 &= [\xi + \alpha(1 - \xi^2)]^2 \\ &= \alpha^2 + 2\alpha\xi + (1 - 2\alpha^2)\xi^2 - 2\alpha\xi^3 + \alpha^2\xi^4 \end{aligned}$$

For this element, the corresponding internal fields are:

$$u = \xi^3 : \rightarrow u^h = \xi$$

$$u = \xi^4 : \rightarrow u^h = \xi^2$$

Therefore, the internal displacement field of the finite element interpolation becomes:

$$u^h = \alpha^2 + (1 - \alpha^2)\xi^2$$

and the resulting strain:

$$\epsilon_x^h = u_{,x}^h = u_{,\xi}^h \xi_{,x} = \frac{u_{,\xi}^h}{x_{,\xi}} = \frac{2(1 - \alpha^2)\xi}{1 - 2\alpha\xi}$$

which can be compared with the correct/applied strain:

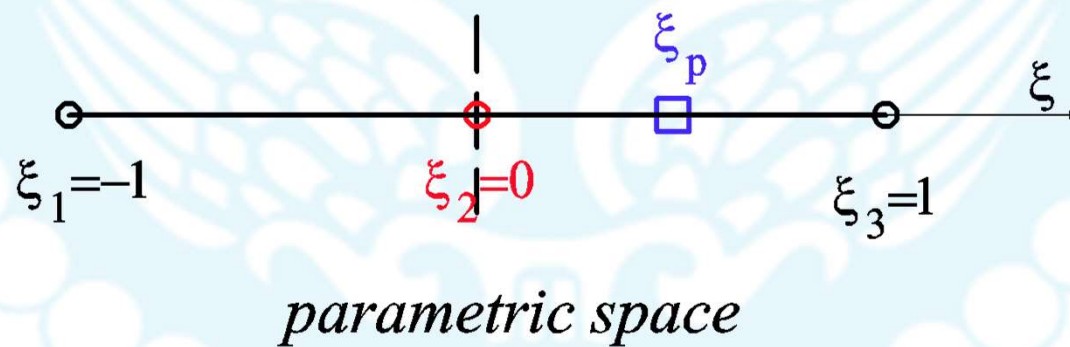
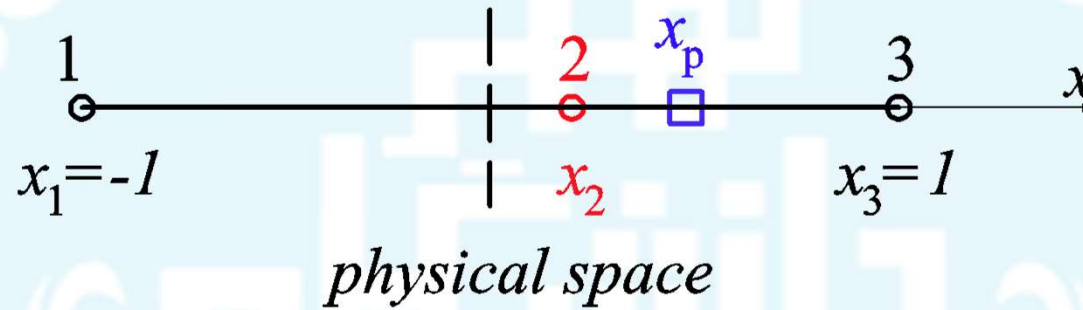
$$\epsilon_x^a = u_{,x}^a = 2x = 2\xi + 2\alpha(1 - \xi^2)$$

To quantify the differences, $\Delta\epsilon_x$ is computed:

$$\Delta\epsilon_x = \epsilon_x^h - \epsilon_x^a = \frac{\overbrace{[-2\alpha(1-3\xi^2)]}^{\text{I}} + \overbrace{[2\alpha^2\xi(1-2\xi^2)]}^{\text{II}}}{1-2\alpha\xi}$$

- It shows that the exact solution is only obtained if $\alpha=0$; the midside node is located exactly at the middle of the element.
- At some points ($2\alpha\xi=1$), the strain becomes infinite at $\xi=1$.
- The first term (I) vanishes for $\xi^2=1/3$.
- The second term (II) vanishes for $\xi^2=1/2$.
- The term (I) is more important than term (II) if $\alpha \ll 0.5$.

Re-examining the element:



While the physical location of the middle node, x_2 , may be anywhere between the end nodes ($-1 < x_2 < 1$), the parametric coordinate of the middle node is always $\xi=0$ regardless of its physical position.

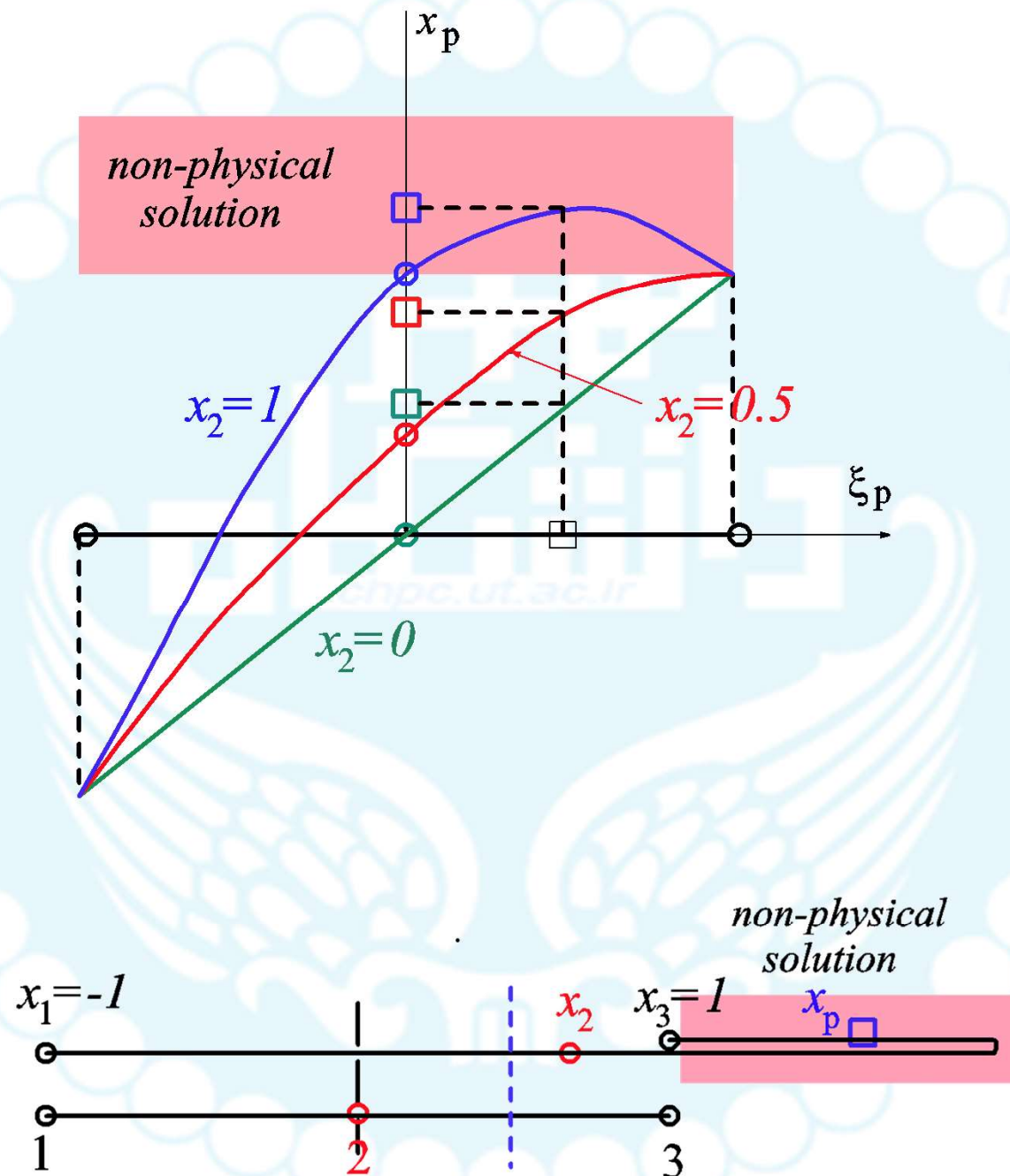
Beginning with the basis function P :

$$P = [1, \xi, \xi^2]$$

The positions x of any node of the element can be described by:

$$x = \xi + (1 - \xi^2)x_2$$

If this equation is plotted for various values of x_2 :

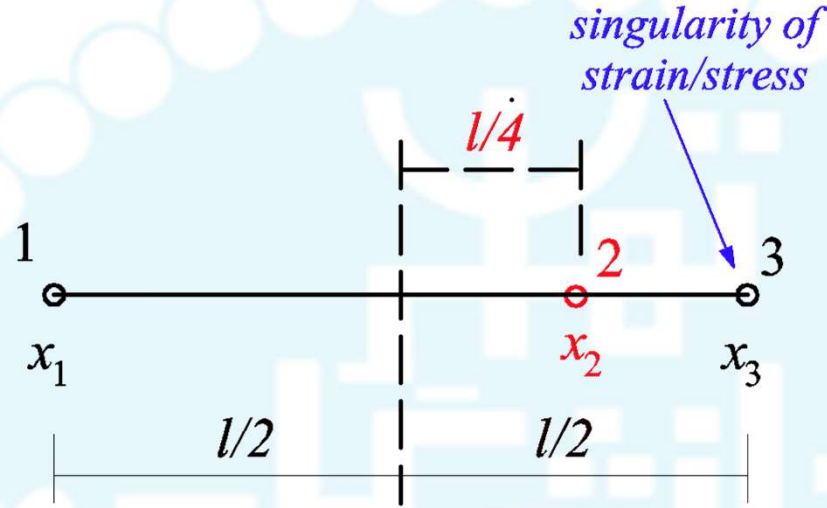


It is observed that, not only in 1D, but also in two and three dimensions, where the edge nodes are located too near the corner nodes, the elements will spill over their boundaries.

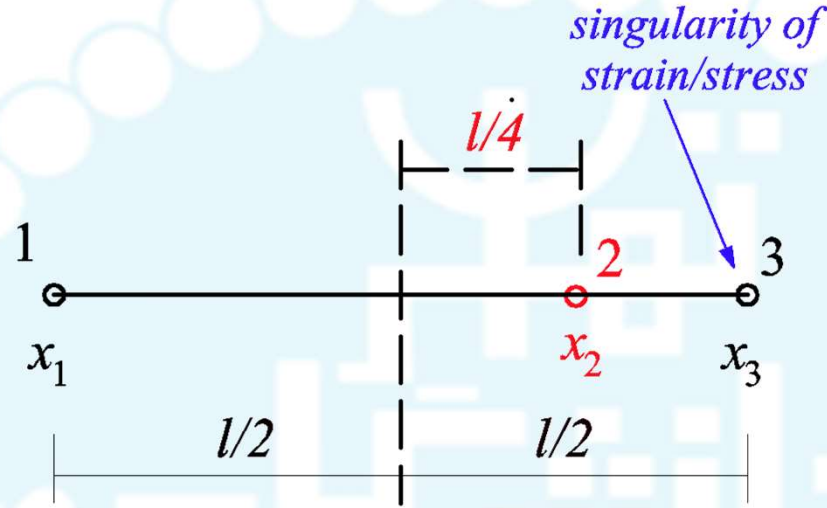
Interestingly, it turn out that placing edge nodes at the so-called “quarter points”, allows the element to simulate a strain/stress singularity at the corner node.

$$\left. \begin{array}{l} x_2 = 0.5 \\ \xi = 1 \end{array} \right\} \xrightarrow{x = \xi + (1 - \xi^2)x_2} x_{,\xi} = 0$$

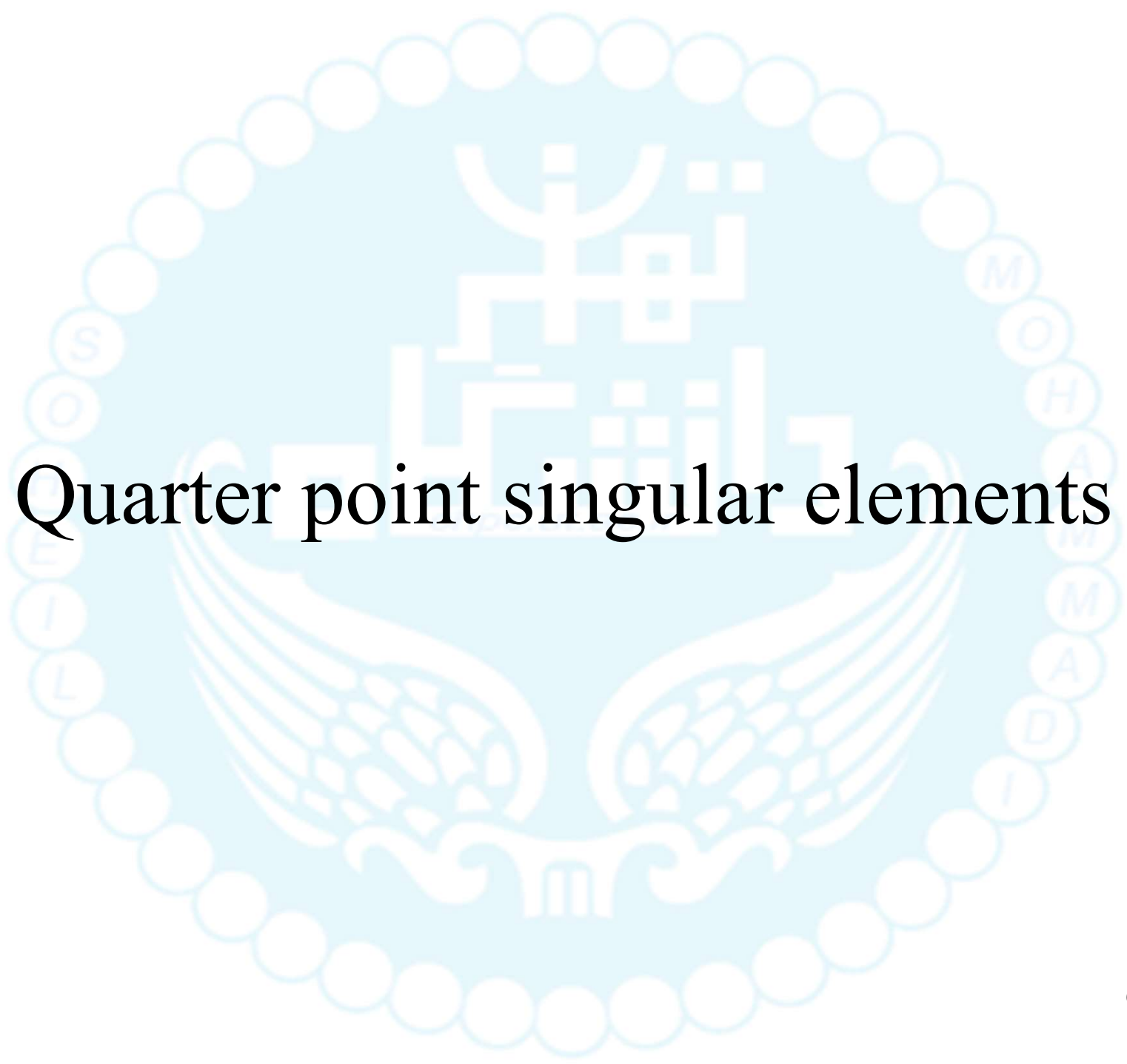
$$\rightarrow \varepsilon_x^h = u_{,x}^h = u_{,\xi}^h \xi_{,x} = u_{,\xi}^h / x_{,\xi} \rightarrow \infty$$



- In special circumstances, a conventional higher order finite element may generate singular strain/stress at a corner node if the corresponding mid-side node is located at the quarter-point position.
- This paves the way for the introduction of singular finite element for fracture mechanics applications.



- A terminal defect in finite element interpolation has led to a significant breakthrough in accurate crack modelling.
- **DO NOT FORGET** that such a singular strain/stress is on the verge of numerical instability of the finite element interpolation.
- Regretably, fracture mechanics references and expert do usually forget to properly acknowledge this important fact.



Quarter point singular elements

Finite element with a singular stress state

In order to the stress state σ to be singular at a point, i.e. a corner node at the crack tip, matrix $\mathbf{B}$ should be singular as the two other terms are constants :

$$\sigma = \mathbf{D}\mathbf{B}\bar{\mathbf{u}}_i$$

For the matrix $\mathbf{B}$ to be singular, the determinant of Jacobian matrix $\mathbf{J}$ must be zero at that point (crack tip):

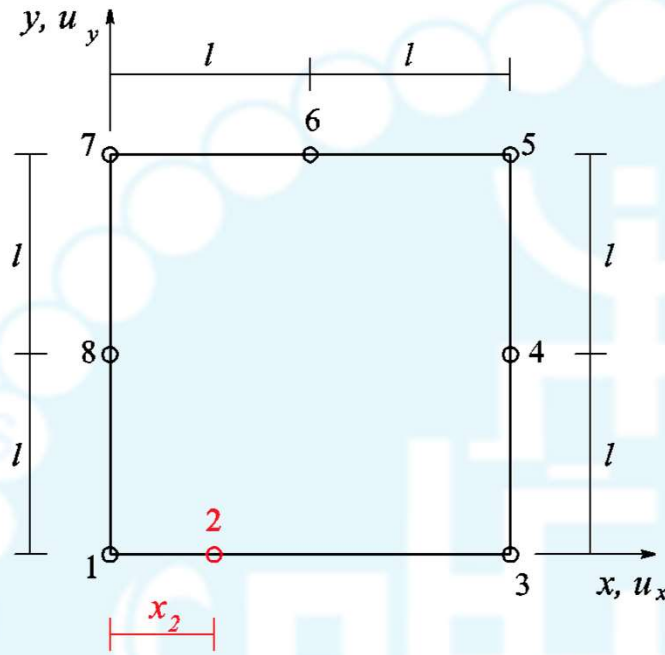
$$\mathbf{B}_j = \begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 \\ 0 & \frac{\partial N_j}{\partial y} \\ \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{Bmatrix}$$

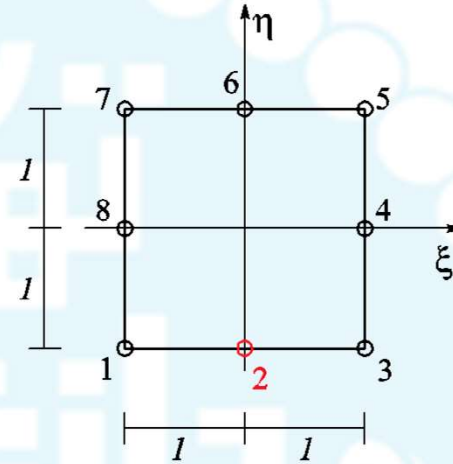
$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad \mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

The determinant of Jacobian matrix $\mathbf{J}$ may become zero, if at least one term on each diagonal terms equals to zero.

This is examined on a rather simplified case of 8-node square element.



physical element



parametric element

$$N_i = \frac{1}{4} (1 + \xi \xi_i) (1 + \eta \eta_i) (\xi \xi_i + \eta \eta_i - 1) \quad , \quad i = 1, 3, 5, 7$$

$$N_i = \frac{1}{2} (1 - \xi^2) (1 + \eta \eta_i) \quad , \quad i = 2, 6$$

$$N_i = \frac{1}{2} (1 + \xi \xi_i) (1 - \eta^2) \quad , \quad i = 4, 8$$

For the straight 1-2-3 edge of the rectangle, both off-diagonal terms of the Jacobian matrix are zero:

$$\left. \frac{\partial y}{\partial \xi} \right|_{\eta=-1} = \left. \frac{\partial x}{\partial \eta} \right|_{\eta=-1} = 0$$

Therefore, if either one of the diagonal terms of the Jacobian matrix equals to zero, the determinant of the Jacobian becomes zero. If the $\partial x / \partial \xi$ is forced to be zero:

$$\begin{aligned} \left. \frac{\partial x}{\partial \xi} \right|_{\eta=-1} = 0 &\rightarrow x_{,\xi} \Big|_{\eta=-1} = 0 \\ &\rightarrow \sum_{i=1}^8 N_{i,\xi} \bar{x}_i = 0 \end{aligned}$$

Using the (derivatives) of shape functions of 8-node element:

$$\begin{aligned}\left.\frac{\partial x}{\partial \xi}\right|_{\eta=-1} &= \sum_{i=1}^8 N_{i,\xi} \bar{x}_i = 0 \\ &= \frac{1}{4}[-1+2\xi+2\xi+1](0) + \frac{1}{4}[1+2\xi+2\xi+1](l) \\ &\quad + \frac{1}{4}[-1+2\xi-2\xi+1](l) + \frac{1}{4}[1-2\xi+2\xi-1](0) \\ &\quad + \frac{1}{2}[-2\xi-2\xi](x_2) + \frac{1}{2}[-2\xi+2\xi]\left(\frac{l}{2}\right) \\ &\quad + \frac{1}{2}[1-1](l) + \frac{1}{2}[-1+1](0) \\ &= \frac{1}{4}[2+4\xi](l) + \frac{1}{2}[-4\xi](x_2)\end{aligned}$$

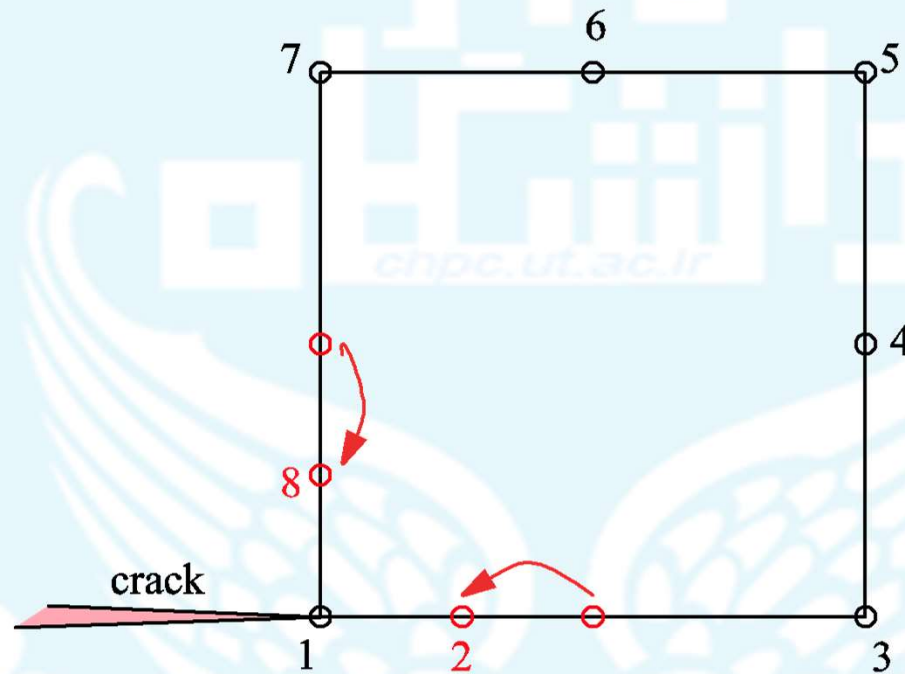
If the final equation is set to zero at the corner node 1:

$$\left. \frac{\partial x}{\partial \xi} \right|_{\substack{\eta=-1 \\ \xi=-1}} = 0 \rightarrow \frac{1}{4}[2 + 4(-1)](l) + \frac{1}{2}[-4(-1)](x_2) = 0$$
$$\rightarrow x_2 = \frac{l}{4}$$

Which means that if the mid-side node 2 is located at the so-called “quarter point”, the determinant of the Jacobian becomes zero, and the stress at corner node will become singular.

Therefore, singularity at the crack tip (located at a corner node), can be achieved by simply moving the mid-side node to its corresponding quarter-point position.

Similarly, for the straight 1-8-7 edge of the rectangle, the $\partial y / \partial \eta$ term is enforced to be zero at the crack tip, which results in moving the node 8 to its corresponding quarter-point position along the 1-8-7 edge.



It should be noted that the stress singularity is, so far, only guaranteed along the 1-2-3 and 1-8-7 lines.

The formulation can be extended to general 8-node elements with curved edges.

While it is clear that the stress becomes singular at the corner node for quarter-point elements, it is yet to be proved that the corresponding stress field inside the element matches the LEFM (Westergaard) stress field; similar order of singularities are achieved.

Again, beginning with the interpolation along the 1-2-3 edge of the square element:

$$\begin{aligned}
 x|_{\eta=-1} &= \sum_{i=1}^8 N_{i,\xi} \bar{x}_i = 0 \\
 &= \frac{1}{2} (1 - \xi^2) (1 + 1) \left(\frac{l}{4}\right) + \frac{1}{4} (1 + \xi) (1 + 1) (\xi) (l) \\
 &= \frac{1}{2} \xi (1 + \xi) (l) + (1 - \xi^2) \left(\frac{l}{4}\right) = 0 \quad \rightarrow \quad \xi = -1 + 2\sqrt{\frac{x}{l}}
 \end{aligned}$$

Similarly, for the displacement field along the 1-2-3 edge:

$$u = -\frac{1}{2} \xi (1 - \xi) (\bar{u}_1) + \frac{1}{2} \xi (1 + \xi) (\bar{u}_2) + (1 - \xi^2) (\bar{u}_3)$$

Which results in:

$$\begin{aligned} u &= -\frac{1}{2} \xi(1-\xi)(\bar{u}_1) + \frac{1}{2} \xi(1+\xi)(\bar{u}_2) + (1-\xi^2)(\bar{u}_3) \\ &= -\frac{1}{2} \left[-1 + 2\sqrt{\frac{x}{l}} \right] \left[2 + 2\sqrt{\frac{x}{l}} \right] (\bar{u}_1) \\ &\quad + \frac{1}{2} \left[-1 + 2\sqrt{\frac{x}{l}} \right] \left[2\sqrt{\frac{x}{l}} \right] (\bar{u}_2) \\ &\quad + \left(1 - \left[-1 + 2\sqrt{\frac{x}{l}} \right]^2 \right) (\bar{u}_3) \\ &= A_1 + A_2 x + A_3 \sqrt{\frac{x}{l}} = A_1 + A_2 x + A_3 \left(\frac{x}{l} \right)^{\frac{1}{2}} \end{aligned}$$

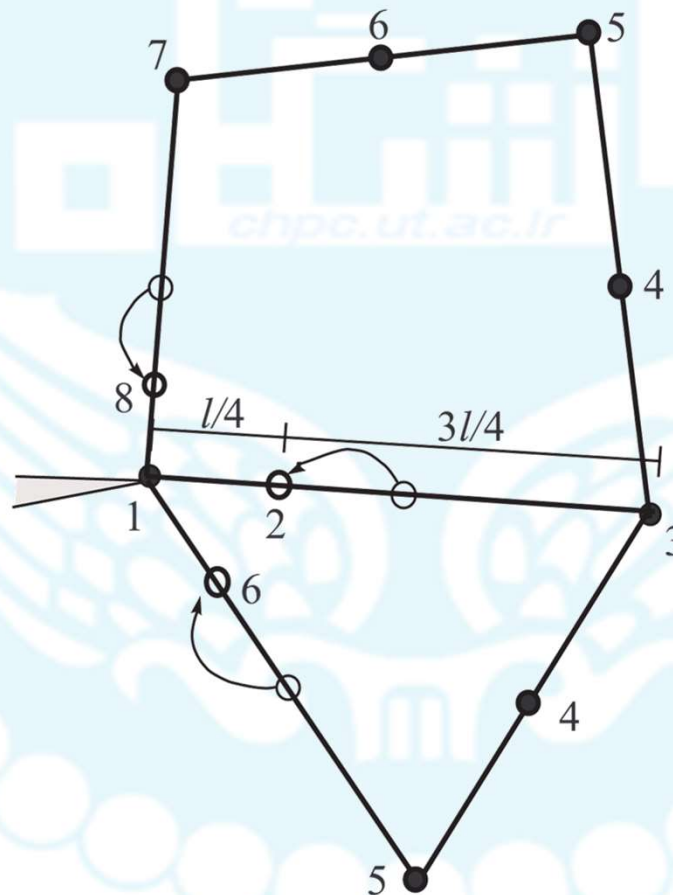
Therefore, the order of singularity for strain (stress) is:

$$\varepsilon = B + Cx^{-\frac{1}{2}}$$

which is similar to the order of singularity in LEFM.

As a result, the quarter-point singular element can be efficiently used for accurate crack analysis. It retains the conventional formulation of the 8-node element, and only requires moving the mid-side node to the corresponding quarter-point position.

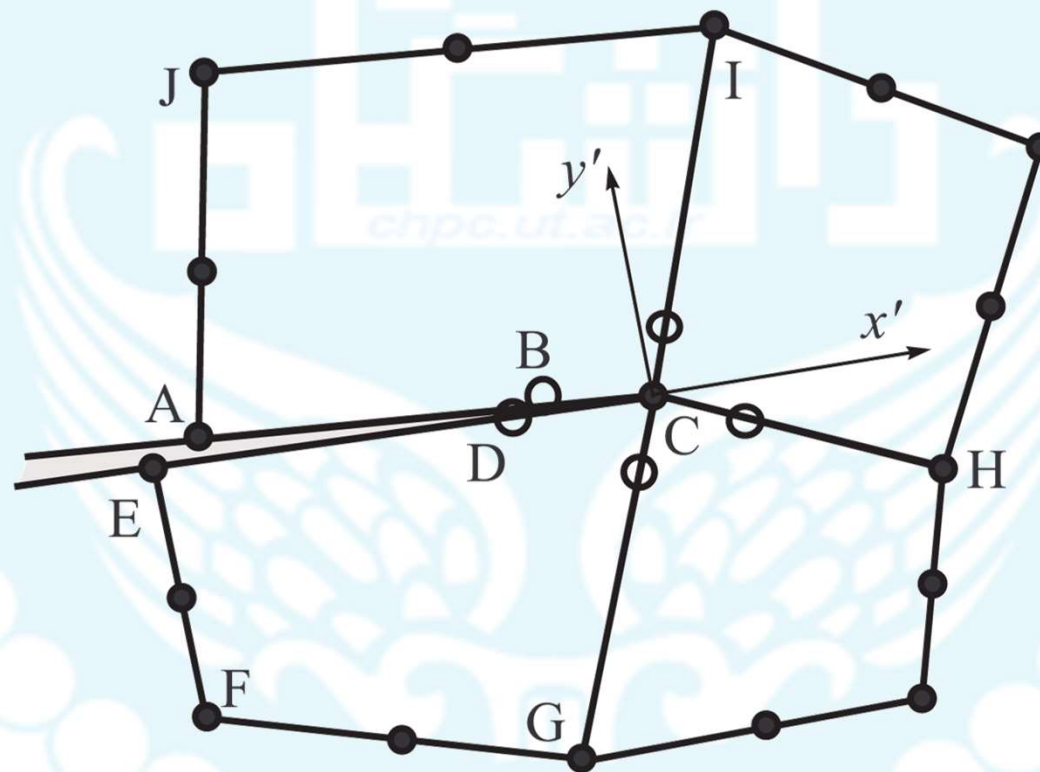
The formulation can be extended to general 8-node elements (with curved edges), as well as the general 6-node elements (with curved edges), if the mid-side nodes are moved to quarter-point positions to ensure singular stress states at the corresponding crack tip corner point.



Singular 6-node/8-node elements:

- 6-node singular elements provide singular stress states along each radial line from the crack-tip corner node inside the element.
- 6-node singular elements also provide more number of elements around a crack-tip node. As a result, singularity is guaranteed along a wider range of radial lines.
- Some references claim that 8-node singular elements also guarantee singularity along each radial line inside the element. This claim cannot generally be verified.

Singular finite elements allow for an efficient and fast approach for evaluation of the stress intensity factors (Shih et al., 1976). The basic idea is to equate the local displacement field in the quarter point singular element with the theoretical solution.



For a general mixed mode problem,

$$K_I = \frac{1}{2} \frac{2\mu}{\kappa+1} \sqrt{\frac{2\pi}{l}} \left[-3u_{y'C} + 4(u_{y'B} - u_{y'D}) - (u_{y'A} - u_{y'E}) \right]$$

$$K_{II} = \frac{1}{2} \frac{2\mu}{\kappa+1} \sqrt{\frac{2\pi}{l}} \left[-3u_{x'C} + 4(u_{x'B} - u_{x'D}) - (u_{x'A} - u_{x'E}) \right]$$

where $u_{x'}$ and $u_{y'}$ are the local displacements of the nodes along the crack in the singular element, with x' aligned with the crack axis.

The extension of the method to the case of more than one singular element, and possible discrepancy of the results obtained from different singular elements, remains unresolved. Nevertheless, the method has remained popular because of the advantage of being exceptionally simple and fast.

